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# Principles of Biosystems Engineering

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# The Biosystems Engineering

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## TOPICS

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-  The **Biosphere**
  -  **Systems** concepts
  -  **Engineering** principles
- 

**T**he field of Biosystems Engineering is emerging in response to such major concerns as environmental integrity, food safety and quality, water security, and natural resource availability. **Biosystems engineering is defined here as the analysis, design, and control of biologically-based systems for the sustainable production and processing of food and biological materials and the efficient utilization of natural and renewable resources in order to enhance human health in harmony with the environment.** In this course, biosystems engineering is introduced to address the large, complex, and time-dependent nature of biophysical systems. However, the principles and examples can be applied and extrapolated to lower-level structures, such as cells, organelles, organs, and organisms.

There is a growing worldwide concern for saving the environment from human abuses and harmful actions. Such activities include irresponsible exploitation of our natural resources, deforestation, pollution of air and water, dumping of non-biodegradable products in our soil, uncontrolled production of ozone-depleting chemicals, acid-rain effects in forests and lakes, and the overall degradation of the environment. Human abuses have slowly destroyed our life support systems. Consequently, the supply of safe food and water is increasingly compromised. The major culprit, among other things, is the lack of understanding of the dire consequences of uncontrolled economic development, unregulated technological advancement, and mismanagement of natural resources.

Coupled with environmental dilemma is the fact that human population is growing exponentially. While human population is expanding, natural resources are declining. According to the World Population Prospects of the United Nations (Archer, et al., 1987) human population will reach close to 8 billion in the year 2020. This implies that there will be more people competing for the consumption of the same amount of natural resources, as illustrated in Fig. 1.1. Air, water, and soil pollution, resource depletion, social chaos, and political upheaval are some of the consequences already demonstrated and are expected to get worse in the future.

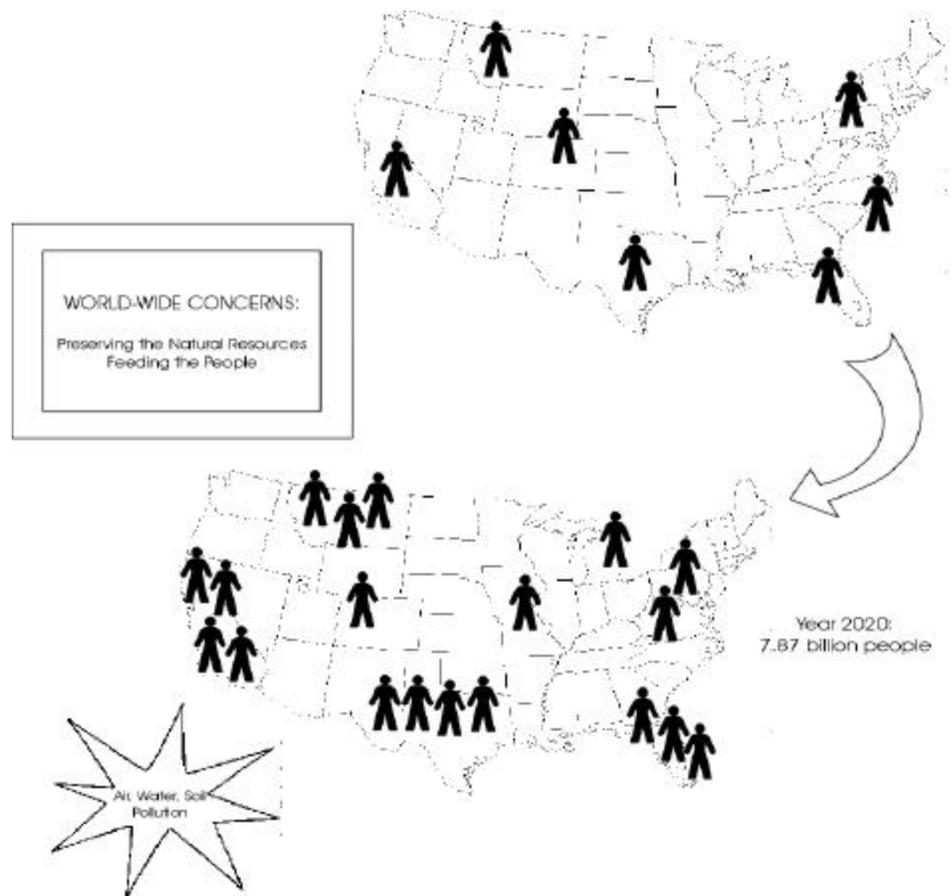


Figure 1.1. Schematic representation of population growth without resource growth, resulting in environmental consequences.

Human-initiated activities have already altered many environmental situations worldwide. For example, cutting down of trees has increased soil erosion; industrial operations have polluted the atmosphere; voluminous human and industrial wastes have resulted in soil and water pollution; the quality of air and water in many cities has reached hazardous level; and changes in heat and water balances in the atmosphere and hydrosphere due to air and water pollution may be factors affecting adverse climatic patterns. Not seen before are the emergence of disease-causing and antibiotic resistant bacteria, such as *Escherichia coli* O157:H7 and *Salmonella typhimurium* DT 104. If the changes in the physical and chemical environments are more extreme than the variations to which human and other living organisms in the ecosystem can adapt, the ecological harmony may be irreversibly disturbed. Therefore, all possible steps should be taken to put an end to the deterioration of the natural environment, as our expression of concern for the future generation. A step toward this end is to analyze problem with global implications in a holistic manner and to propose solutions that reflect the consideration of the whole biosystem. It is the responsibility of the present

generation to maintain a balance among food supply, economic development, and environmental protection in order to provide for, and ensure the survival of, the future generations.

## 1.1 The Biosphere

While the concept on the environment involves a complex system of interacting living and non-living components encompassing the whole universe, for practical and obvious reasons, let us begin with the biosphere as a component of planet Earth. The biosphere is the space where biotic and abiotic worlds meet, at the overlap and interface of the three major spheres: atmosphere, lithosphere, and the hydrosphere (Archer et al., 1987), as illustrated in Fig. 1.2. It is the common ground shared by humans and other living organisms, constantly interacting with one another (Fig. 1.3). These living organisms also exchange matter and energy with their environment. It is in this sphere where the most pressing problems of the environment exist. In general, we shall refer to a system structure in the biosphere involving a biological component as a biosystem. In particular, the biosystem is defined here as any form of organization which is made up of living and non-living components, interacting and interconnected as to achieve a unified purpose, specifically with respect to food production, environmental preservation, economic development, and technological advancement.

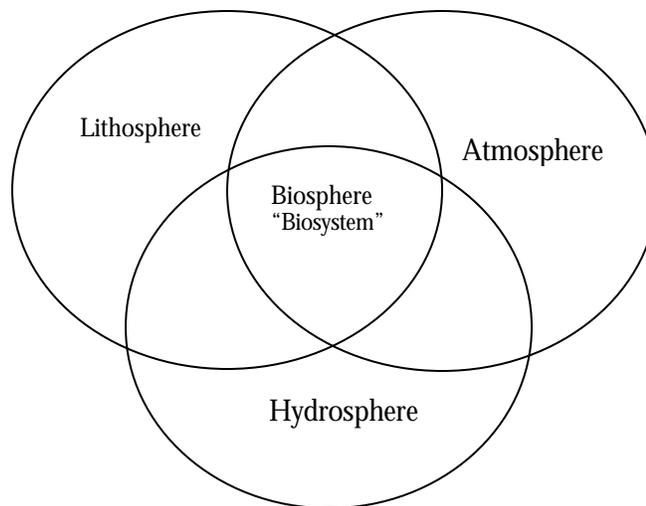


Figure 1.2. Schematic representation of the biosphere as an interface of the atmosphere, lithosphere, and hydrosphere.

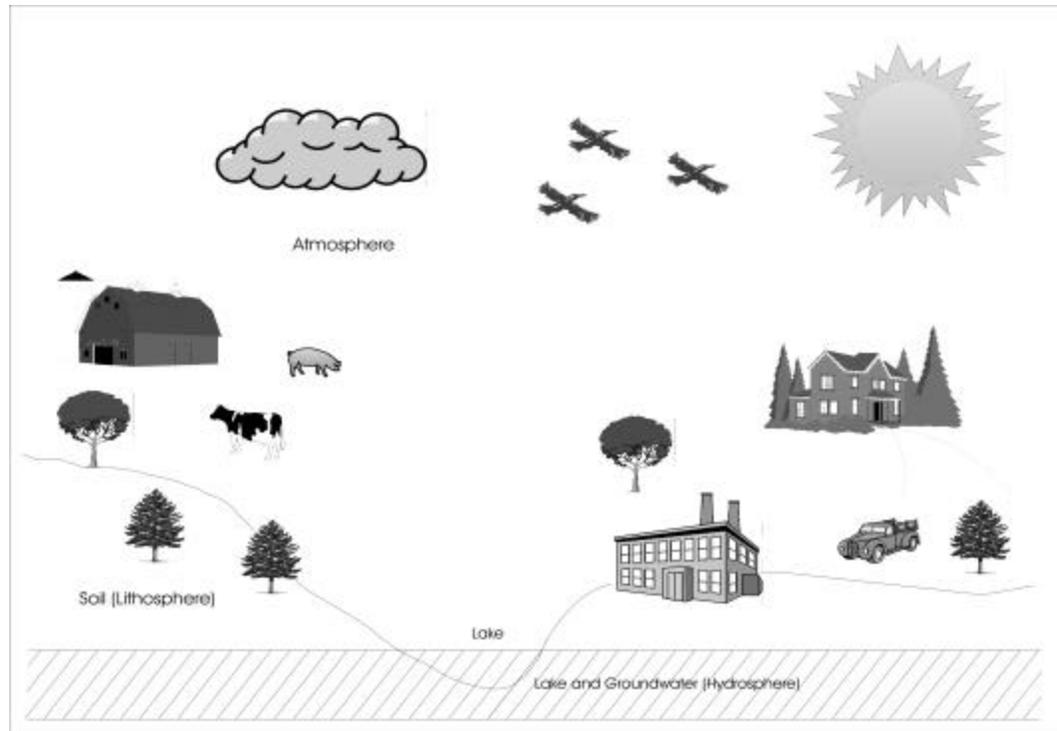


Figure 1.3. Schematic of the biosystem in the biosphere.

## 1.2 Systems Concepts

**System.** The word “system” can be defined as anything formed of parts or components placed together and interconnected to make a regular whole working as if one body or entity as it relates an input to an output, or a cause to an effect. There are at least four concepts in this definition. First, a system is made up of **components** or **subsystems** which have defined relationships. Second, each of these components are **linked** in such a manner that the output of one is an input to the other. Third, the successful operation of one component depends upon the other (**unity**). And fourth, system components are interconnected to form one body or entity in order to achieve its **purpose**. A plant is a good example of a biological system. Plant growth is orchestrated in its internal mechanism as to reproduce itself. Under favorable conditions, a corn plant will bear corn grains (not rice nor wheat grains). The photosynthetic process of converting solar radiation into carbohydrates and finally biomass is a series of interconnected transformations. When photosynthesis fails, biomass will not be produced. A car, an airplane, a computer, a microscope, a dog, a tree, a house, a population, a person, a bacteria, and a cell are each an example of a system.

Associated with the word “system” are terms that need clear understanding and comprehension. These words are input, output, parameters, state variables,

boundary, and environment. There are two kinds of input: controllable and exogenous. Similarly, there are two kinds of output: desired and undesired. These definitions are presented as follows.

**Controllable Input.** The controllable input variables are materials or energy which are required to bring about the desired system output. These variables can vary with time. For example, water is a material input in soil-plant systems, animal production systems, and river or lake systems. The volume of water flowing into a river may vary during the day. Food intake is a controllable input to the body.

**Exogenous Input.** The exogenous input variables are materials or energy, which influence or affect the biosystem but the biosystem cannot affect them (at least for the system under consideration). For example, solar radiation, air temperatures, and rainfall are exogenous input to people, forest, crop, urban, and economic systems.

**Desired Output.** The desired output variables are the transformation product of the material input and the system processes (accounting for technologies) through the use of energy and labor. For example, forage and grain are desired outputs of the corn production system; milk, meat, eggs, and fur are desired outputs of the animal system; profit is a desired output of a farm system; potable water is a desired output of a regional system.

**Undesired By-Products.** The undesired by-products are the undesirable results as the biosystem functions to produce the desired outputs. For example, nitrate leaching is an undesired by-product of crop production system; phosphate runoff is an undesired by-product of animal production system; water pollution is an undesired by-product of an industrialized economy.

**State Variables.** The state variables summarize the status of the system. Knowing the state variable (S) at any initial time  $t_0$  and the input function (I) at time  $t_0$ , together with the equations describing the dynamics of the system, it should be possible to know the state of the system at any time  $t_1$  where  $t_1 > t_0$ . That is,

$$I(t_0) + S_0(t_0) \rightarrow S_1(t_1)$$

State variables can be classified into two: feasible and infeasible. The feasible states are variables that satisfy the constraints of the system and therefore are valid information in assessing the status of the system. The infeasible states are variables that violate at least one constraint. For example, the weight of leaves, stems, grain, and roots are state variables of a plant system; the amount of biomass and milk are state variables of a dairy system.

**System Parameters.** System parameters are factors, which determine the initial structure and condition of a biosystem. In mathematical equations, these are constants representing technology or information. Parameters are differentiated

from state variables in that, for deterministic systems, they do not change with time during the operation of the system.

**System Boundary.** System boundary is the separation (real or imaginary) between the system and the environment. For example, the physical boundary of a household system may be the house structure itself, that is, everything inside the house belongs to the system; everything outside belongs to the environment.

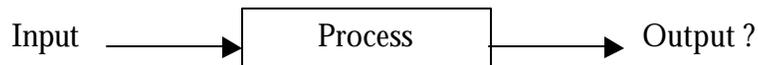
**Environment.** For any given biosystem, there is an environment. This environment is the set of all objects, factors, and influences outside the boundary of the system. All signals from the environment crossing the boundary into the system must be one-way direction, that is, the signal may affect the system but the system output should not affect the environment to the extent that it would modify the signal (Eisen, 1988). The environment may occur in the following forms:

1. Natural environment -- For a biological (e.g. crop production) system, the natural environment may include solar radiation, rainfall, ambient temperatures, and wind speed.
2. State-of-resource-and-technology environment -- Formulation and structuring of a crop production system may be affected by the type of irrigation to be employed, or the crop variety to use, or the fertilizer management to practice. It is also affected by the availability of production inputs, accessibility to markets, etc.
3. State-of-knowledge environment -- Knowledge of the processes affect the formulation and synthesis of a biosystem. When there is no clear understanding of the biosystem, a less efficient and less sustainable management approach is likely to be used. For example, our lack of deeper understanding on the extent and ill-effects of nitrates and pesticide residues in groundwater contributed to the neglect of sustainable practices in manufacturing industries, farms, golf courses, household gardens and lawns, and other operations.
4. Institutional and Social Environment -- The institutional, organizational, and social structures, such as government laws, regulatory bodies, lobby groups, commodity associations, social customs, personal preferences, and manpower skills, may influence the evaluation of objectives and the structuring of the biosystem. For example, certain commodities dominate the market because of trade agreements.
5. Economic Environment -- Input costs, product prices, marketing costs, and other economic factors affect the formulation, structuring and synthesis of a biosystem. For example, cheaper inputs are likely preferred over more expensive materials.

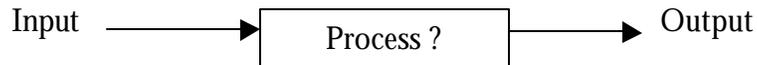
### 1.3 Engineering Principles

Basic engineering skills include analysis, design, and control.

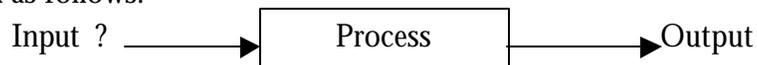
**Analysis.** Analysis is the process of finding the solution (output) of a specified system process, given a description of the system inputs. For example, a nutritious food (input) fed into a healthy body (system process) could result in muscle gain (output). An analysis can be done to determine the amount of muscle gain for every gram of food intake in the same body structure. In the analysis, the amount of food intake may be varied and the gain observed. Schematically, analysis can be represented as follows:



**Design.** Design is the specification of the system process in order to match specific input to desired output. Continuing on our body example, two people given the same amount of food may gain weight differently due to differences in their body structure. To compensate for the difference, one person may be advised to get more exercise in order to achieve the same weight gain. The concept of design can be diagrammed as follows:



**Control.** Control is the specification of inputs in order to achieve desired outputs given a description of the system process. An individual may be prescribed a set of food to achieve a desired weight gain. In schematic presentation, the relationship can be seen as follows:



Systems analysis is the application of organized analytical modeling techniques appropriate for explaining complex, multivariable systems (Vaidhyanathan, 1993). Our efforts in this course shall be directed towards the application of modeling techniques to understand important dynamic phenomena for the continuing stability of the biological system. The study shall be guided by the principles of growth, conservation of mass, cybernetics, stability, and sustainability.

Growth is the principle of gradual development of living matters toward maturity. It is a process involving an increase in size, weight, power, wisdom, and many other factors. Decay is the antithesis of growth. A gradual decline in strength, soundness, health, beauty, and prosperity is part of decay. It is also a process involving decomposition and rotting. Chapter 4 will cover extensively the principle of growth.

Conservation of mass (as well as energy and momentum) is the principle that matter cannot be created nor destroyed during a physical or chemical change. Within a closed system, the rate at which a substance increases or decreases is due to the rate at which the substance enters from the outside minus the rate at which it leaves. However, while matter cannot be destroyed nor created, it can be transformed, transported, and stored. Conservation equations are "bookkeeping" statements: they serve to keep track of all substances within the boundary of the system, including the initial conditions. More discussions and illustrations will be presented in Chapter 5.

Cybernetics is the science of how systems are regulated. The cybernetic structure can either be open-loop or closed-loop. An open-loop biosystem can be partitioned into two major structures: the *bio-control* and *bio-process* structures, as illustrated in Fig. 1.4. The *bio-control* structure receives the external input signal which represents a goal, an objective, or a reference of what is desired to be achieved. The *bio-process* structure takes material resources and/or energy and acts on them to produce the outputs. This structure is subject to disturbances, which may cause variations in the output and in the process itself. In an open-loop biosystem, the actuating signal can be altered at any time based on the objective of the output and all other prior knowledge about the process. This actuating signal is not influenced by the output of the system at any time.

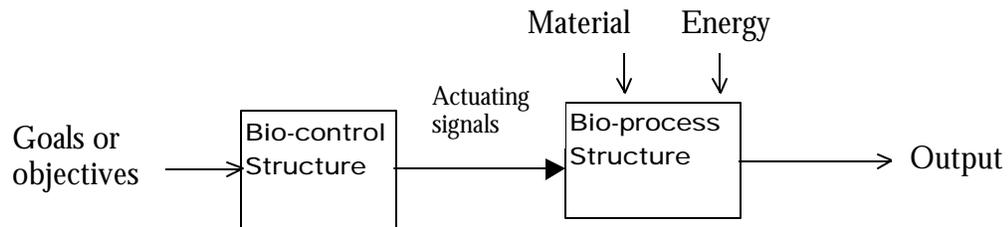


Figure 1.4. A generalized structure of an open-loop biosystem.

However, when the output affects the input signal, then the system is called a closed-loop biosystem as shown in Fig. 1.5. In this case, a *bio-sensor* or control measurement is present, which consists of a measuring device to record output signals and convert them into internal input signals to the *bio-control* structure. The internal input signals provide the *bio-control* structure with information on what is happening in the *bio-process* structure. The engineering of the biosystem consists of defining the nature of the *bio-control* structure, the character of the *bio-process* structure, and the description of the *bio-sensor* so that the outputs agree with the objective for which the biosystem is being directed.

The biosystem structure in Fig. 1.6 can be broadly represented by a *control* structure that is human-dominated interfacing with the *real* structure (natural, biological, and physical) as shown in Fig. 1.6 (Alocilja and Ritchie, 1992). The decision maker resides in the *control* structure. It is in this structure that objectives are defined and subsequently, decisions on controllable inputs are made in order to

achieve the desired outputs, manage the undesired by-products, and control the transformation processes. The decision maker in the *control* structure determines how the processing components in the *real* structure are interconnected. The *real* structure, on the other hand, is the object of control. It is composed of the natural, biological, and human-made physical systems. This structure is multi-level and requires an integration of multidisciplinary sciences. Constraints in the *real* structure are fed back to the *control* structure. The *control* and *real* structures are, therefore, highly coupled: the objectives as defined by the *control* structure, determine the interconnections of the processes in the *real* structure, while the statement of objectives in the *control* are constrained by the boundary conditions existing naturally, biologically, and technologically in the *real* structure. Examples 1.1 and 1.2 provide illustrations of open-loop and closed-loop systems. Chapter 4 will illustrate further the concept.

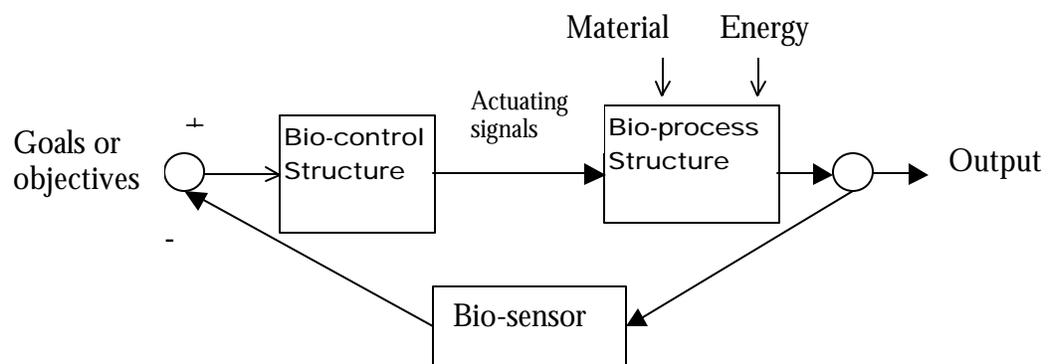


Figure 1.5. A generalized structure of a closed-loop biosystem.

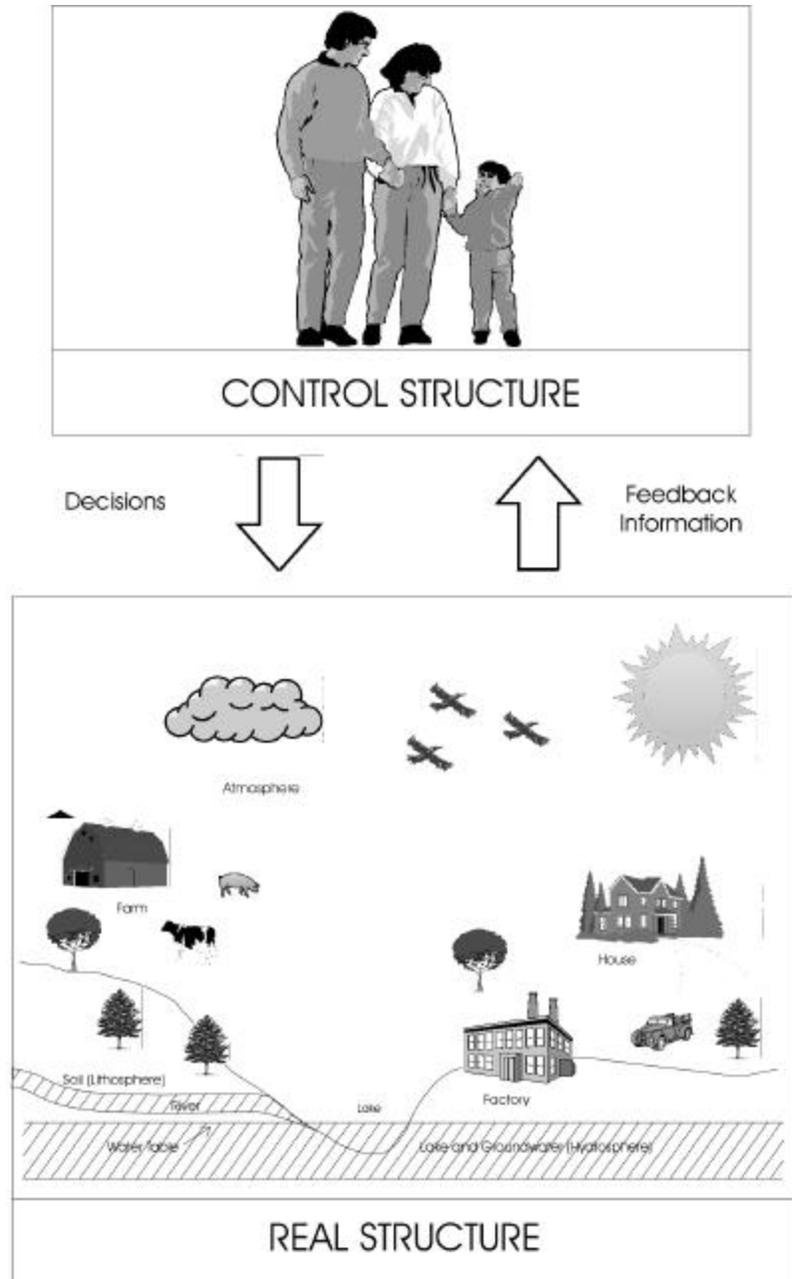


Figure 1.6. Schematic diagram of a human-controlled structure.

The important factor in a controlled system is the presence of feedback, which permits the system to compare what the effector is doing with what it is supposed to be doing. A feedback loop exists if variable  $x$  determines the value of variable  $y$  and variable  $y$  in turn determines the value of variable  $x$ . In living systems, examples of effectors are muscle cells, cells which secrete various

chemicals into the bloodstream, or cells which transport substances from part of the body to another (Jones, 1973).

An example of an open-loop feedback system would be growing a corn plant to which water and fertilizer are regularly added without regard to the condition of the plant. Wilted or flooded, water is added regardless. The adding of water and fertilizer is not influenced by how much the plant needs at any given time. In this case, the flow of information is in one direction; there is no return flow or “feedback” of information as to the actual situation of the plant. Figure 1.7 illustrates the concept graphically.



Figure 1.7. Schematic diagram of an open-loop corn plant culture.

Suppose the controller varies the amount of water according to how the plant looks at any point in time. If the plant looks wilted, water is added; if the plant looks “sick”, fertilizer is added; if the soil feels saturated, water is suspended. In this instance, there is an information flow from the plant, which the controller uses to regulate the input of water and nutrients. In this case, the system is closed-loop feedback and illustrated in Fig. 1.8.

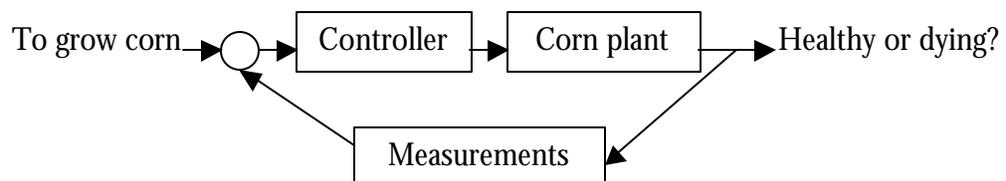


Figure 1.8. Schematic diagram of a closed-loop corn plant culture.

Ecosystems act as open loop systems because they receive radiant energy from the sun and moisture from rain and snow which they cannot influence by feedback. Living things in the ecosystem are influenced by temperature, which the system has no control over. However, ecosystems also act as closed-loop systems because there are many ways whereby they can regulate the flow of materials or the effect of various physical impacts on them.

Negative feedback regulation is often referred to as deviation-counteracting feedback, or homeostasis (DeAngelis et al., 1986). Many ecological systems have generally been thought of as homeostatic systems. Although there is still disagreement on the precise interactions responsible for the negative feedback and on how tightly it regulates various ecological systems, there is some consensus that negative feedback regulation is occurring at least to the degree that it normally

keeps populations and communities from going completely out of control, although it may not always be strong enough to prevent sizable fluctuations. Positive feedback occurs when the response of a system to an initial deviation of the system acts to reinforce the change in the direction of the deviation.

When a population or ecological community is perturbed slightly from equilibrium, the balance of negative and positive feedback mechanisms in the system tends to counteract the perturbation and restore the system to equilibrium (if this equilibrium is inherently stable). Neither positive nor negative feedbacks manifest themselves individually in a conspicuous way. However, when the population or community is driven far from equilibrium (by a temporary unusual environmental condition, for example), specific positive or negative feedback may become very obvious. The system could be driven into a regime where homeostatic forces no longer operate. Positive feedback loop may act to amplify the deviation, perhaps driving the system to extinction. On the other hand, other positive feedback loops may be activated that rescue the population from extinction and allow it to increase to the point where homeostatic forces restore it to equilibrium. Pest or epidemic outbreaks and the survival of rare species in environments of scattered habitat islands are all problems that involve positive feedback in important ways.

The existence of a flow of information from the system “output” (in the example, the plant’s appearance) to the controller that regulates the input to maintain a stable set point (a healthy plant), is a commonly accepted emblem of a cybernetic system. The systems considered in this chapter are more general than this in two ways. First, the feedback will not always be purely information flow, but may be a change in biomass, energy, or material that only incidentally conveys information affecting the system input. Secondly, the feedback in question will not necessarily maintain the system at a stable point, but may cause the system to change from one state to another.

### **Example 1.1. A rider-bike system**

Consider a system structure of a person riding a bicycle. The rider-bike system can be decomposed into several components in a simplified manner, as shown in Fig. 1.9. The objective of the rider is to keep the bike in the center of the bike lane. The output is the actual path of the bike. An error in the difference between the objective and output are detected visually. This error detection activates the brain. The brain transmits signals to the rider’s arms and muscles which control the steering handles and the wheel directions.

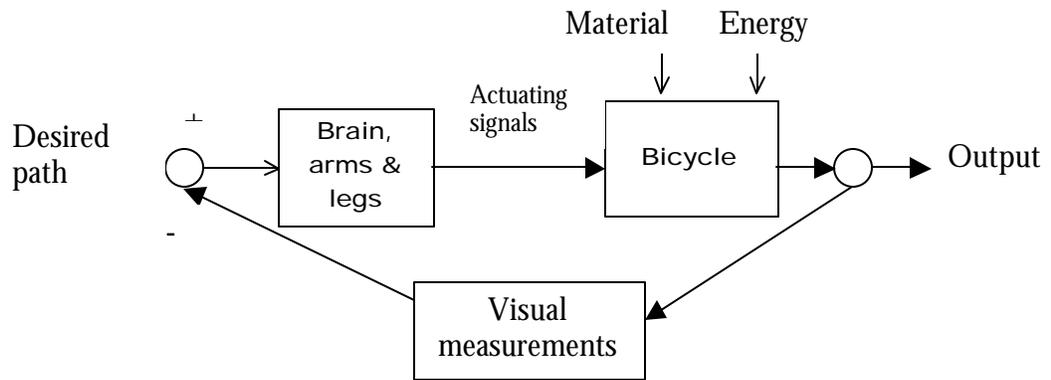


Figure 1.9. A schematic diagram of a rider-bike control system.

### Example 1.2. A Plant-water system

Consider a system of a potted rose that is watered daily. The objective is to grow a beautiful, flowering rose bush. If water is added without regard to the condition of the plant, the system is an open-loop system. However, if water is added only when the plant needs it, or if the amount varies to meet the moisture requirement of the plant, then the system is a closed-loop system—the output (condition of the plant) affects the input (frequency and amount of water applied to the pot).

Related to cybernetics is the idea of homeostasis or stability of system behavior. Stability is concerned with how the system handles perturbations or disturbances. The basic notion of system stability is this: if a system initially at rest is disturbed, does it gradually return back to rest or does it wander away? If the system returns back to its rest position, then the system is stable; if it wanders away, then the system is unstable.

Sustainability is the philosophical paradigm that addresses economic growth and development within the limits set by ecology in the broadest sense—by the interrelationships of human beings and their works, the biosphere, and the physical and chemical laws that govern it (Ruckelshaus, 1989). The World Commission on Environment and Development (Haines, 1992) defined sustainable development as the “development that meets the needs of the present without compromising the ability of the future generations to meet their own needs.” According to Haines (1992), sustainable development has become the quintessential paradigm for addressing the worldwide dilemma of advancing our economic development while protecting environmental quality not only for this generation but future ones as well.

This book shall bring together many disciplinary sciences and study the interrelationships among them so that we may be able to learn, understand, and recognize the interconnections among food security, technological advancement, environmental integrity, and economic development. In so doing, biosystems engineers hope to be able to design and engineer a sustainable biosystem that will be compatible with, and able to accommodate, **change, complexity, and growth.**

## Exercises

1. *Impacts of High Population.* Study the schematic diagram of Fig. 1.1. Identify the implications and effects of over-population. Note that people need three basic things: food, clothing, and shelter.
2. *Learning systems concepts from the headhunters.* In a remote forest, there happened to be three headhunters and three strangers arriving in the same place at the same time. Considering that there was a one-to-one ratio among them, peace prevailed. All were seeking to cross a river in a boat that can hold at most two people and can be navigated by one or two individuals. Due to their headhunting practice, the headhunters could not be allowed to outnumber the strangers at any time and at any place. All were able to cross the river in five and one-half round trips without mortality. How did they do it? Let the bank where the people were originally be designated as bank 1; the bank across be designated as bank 2. Identify the system boundary, system parameters, input, output, and state variables. Identify the feasible and infeasible states.
3. *Analysis and design of the International Center Food Court.* The International Center Food Court (ICFC) is the place to be during mealtimes. Conduct an analysis and suggest design improvement, if necessary, so customers are satisfied.
4. *Systems definition in lake pollution.* Pollution occurs when high amounts of nutrients, through soil erosion and waste disposal, end up in the lake. Nutrient accumulation favors the growth of certain tiny marine organisms called phytoplankton. These plankton blooms appear as patches on the surface of the water. During this process, oxygen in the water is depleted and other living organisms are affected. In such a condition, identify the system boundary, the environment, system parameters, input, output, and state variables of a lake system.
5. *Understand systems concepts in the classroom.* Consider BE 230 class as the system. For a given lecture hour, identify the controllable and exogenous inputs, state variables, desired output, undesired by-product, design parameters, and the different kinds of environment. Classify input and output as either material or energy.
6. *Fish tank.* Given a fish tank (12 ft long, 6 ft wide, 6 ft tall) with 5 fish in it (walls are the boundary), identify the following in **quantitative** terms: Controllable input (how much food is needed of each per day?), desired output, undesired by-product, state variables, system parameters, and environment. Illustrate (draw) your system.

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## Systemic Properties of Biological Systems

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### PRINCIPLE

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Universal characteristics among systems.

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### TOPICS

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- ✓ Hierarchy
- ✓ Modularity
- ✓ Network
- ✓ Wholeness
- ✓ Purpose
- ✓ Open system
- ✓ Feedback
- ✓ Stability
- ✓ Reproduction
- ✓ Stochasticity

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One major systems concept is that there is universality of characteristics among systems. That is, any biological system will possess certain properties that will characterize it as a system. The following biological system properties will be covered in this chapter, namely: hierarchy, modularity, network, wholeness, purpose, open system, feedback, stability, reproduction, and stochasticity. Order to which the properties are enumerated has nothing to do with their relative importance. In some references, these properties are lumped together or are separately mentioned. Each property is described by one or more sub-properties.

#### Hierarchy

A biological system at one structural level can be partitioned into subsystems of a similar class to form a lower level structure.

Consider a national food production system, as illustrated in Fig. 2.1. The system at the national level can be partitioned into political boundaries, such as region, state, county, city, and township, or economic sectors, such as agriculture, natural resources, food, industries, communities, etc. Each of the political units can be decomposed into production units, such as farming, processing, marketing, distribution, and consumption systems. A dairy farming system can be partitioned into the soil, crop, irrigation, harvesting, storage, feed processing, feed allocation, animal herds, waste collection and storage, waste utilization, product packaging, and product distribution. A plant system can be decomposed into the roots, leaves, stems, panicles, and grains. Each of these organs can be partitioned into cells, the cells into molecules, molecules into proteins, proteins into amino acids, amino acids into base pairs, and so on.

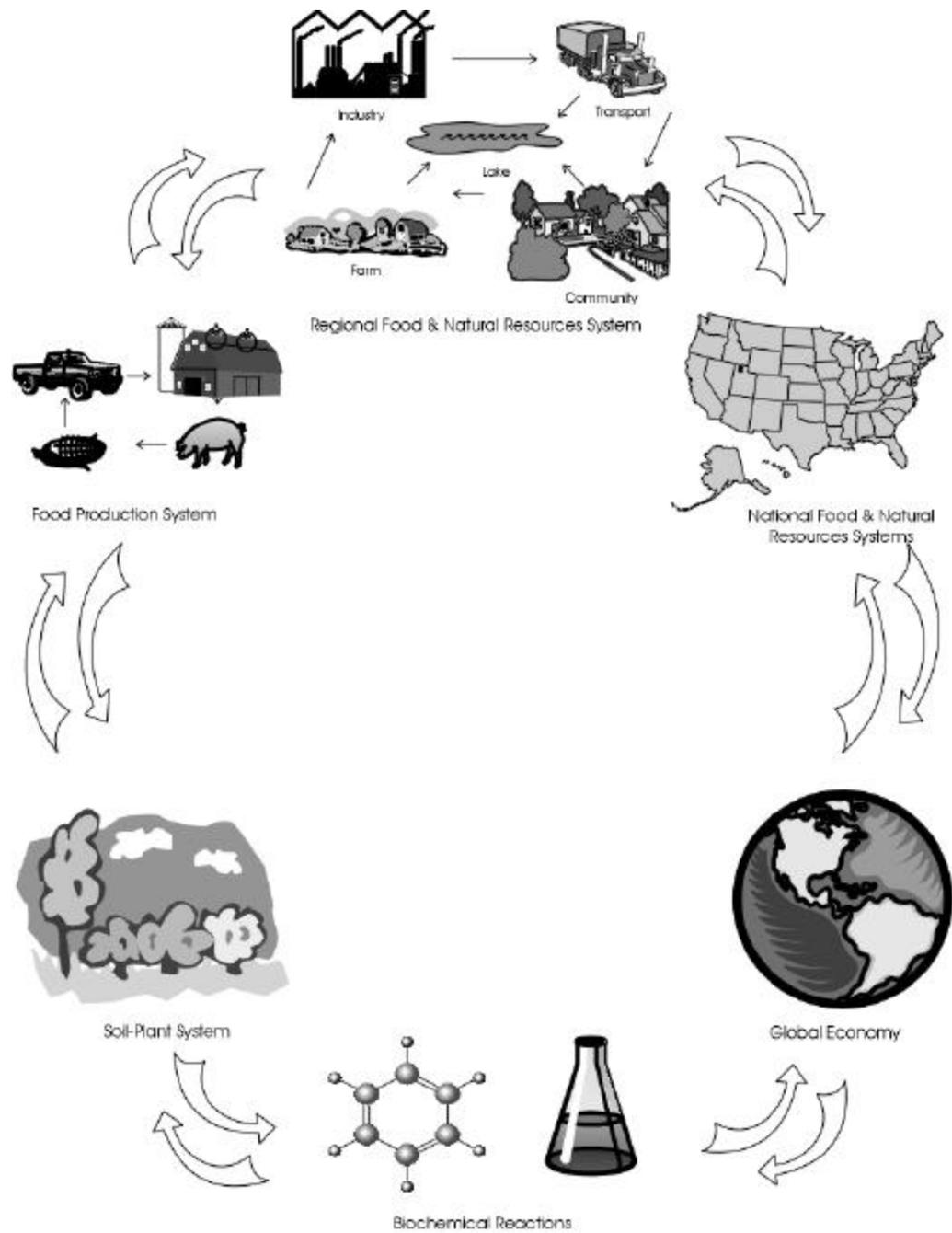


Figure 2.1. Schematic diagram of a food production system in hierarchy.

A biological system may be a component of a larger system.

A biological system, with its inputs, processes, and outputs, is always part of a system greater in scope or hierarchy. For example, the crop system is part of the farm system; the farm system, part of the regional food production system; the regional food-production system, part of the national economy; and so on. Political systems follow similar hierarchy.

The components of a biological system are themselves systems.

The components of an agroeco-regional system may include several food production systems while each production unit is a system in itself. A soybean production system may include a soybean crop, a rotary tillage unit, a combine, and a storage facility. Furthermore, each of these components has the properties of a system. They can be partitioned into lower level components.

There are several functional hierarchy of analysis which are important in distinguishing one biological system from the other. From a biological point of view, these functional levels are (Archer et al., 1987): (1) macromolecular, the activity of biologically active large molecules; (2) cellular, the level at which the living unit is the cell; (3) whole organism or species, which is composed of living cells; (4) population, a cluster of living organisms belonging to the same species; and (5) ecosystem, the interaction and interconnection of living and non-living populations and components of nature on a given space in the biosphere. Eco-regional system, consisting of several ecosystems, can be added to this functional hierarchy. An example of an eco-regional system is a regional rice-production system, which may include irrigated lowland ecosystems, rainfed lowland ecosystems, and upland ecosystems.

The systems forming at the lower level of the hierarchy interact with the systems at the higher level through the flow of materials and energy.

The phytoplankton, fish, microorganisms, aquatic plants, and water comprise the lake system. At a higher level of analysis, the lake can be viewed as a single entity (say, irrigation source or discharge sink) interacting with crop and animal production systems, processing industries, and urban/rural population in a geographic location to form a watershed system. Several watersheds form an eco-regional system, such as the Great Lakes region. By repeating this process, it is possible (at least conceptually) to move systematically from the lake (micro) to a regional or national level of analysis (macro).

## Modularity

A biological system can be partitioned into natural, biological, and artificial.

Some examples of natural systems are the soil, lake, river, and watershed. The plants, animals, insects, weeds, fungi, and other pests may be included in the biological systems. Artificial systems may come in the form of tillage, harvesting, transportation, food processing, and manufacturing facilities.

Each of the components in a biological system can be characterized as discrete.

In a discrete system, there is a clear boundary between the system and its environment. In a soil-crop system, the soil, plants, and production machinery have well-defined boundaries between the system and the environmental factors such as solar radiation, temperature, rainfall, other plants, animals, human intervention, etc. (Fig. 2.1). In a lake system, the plankton, fish, microorganisms, and aquatic plants occur in discrete units.

The number of components constitutes the complexity of a biological system.

A system with 3 components may be easier to analyze than a system with 10 components. For example, a monoculture crop system may be less complex to manage than a multi-crop system.

Each component of a biological system is viewed as performing one or a combination of the three basic processes: transformation, transportation, and storage.

Material transformation is the process of converting a material input of one form into an output of another form with the application of energy and/or human labor. Transport process is the transfer of a material from one location to another without affecting the form or composition of the material. Storage is the flow of material from input to output without affecting the composition of the original form but flows at different points in time. A schematic representation is shown in Fig. 2.2.

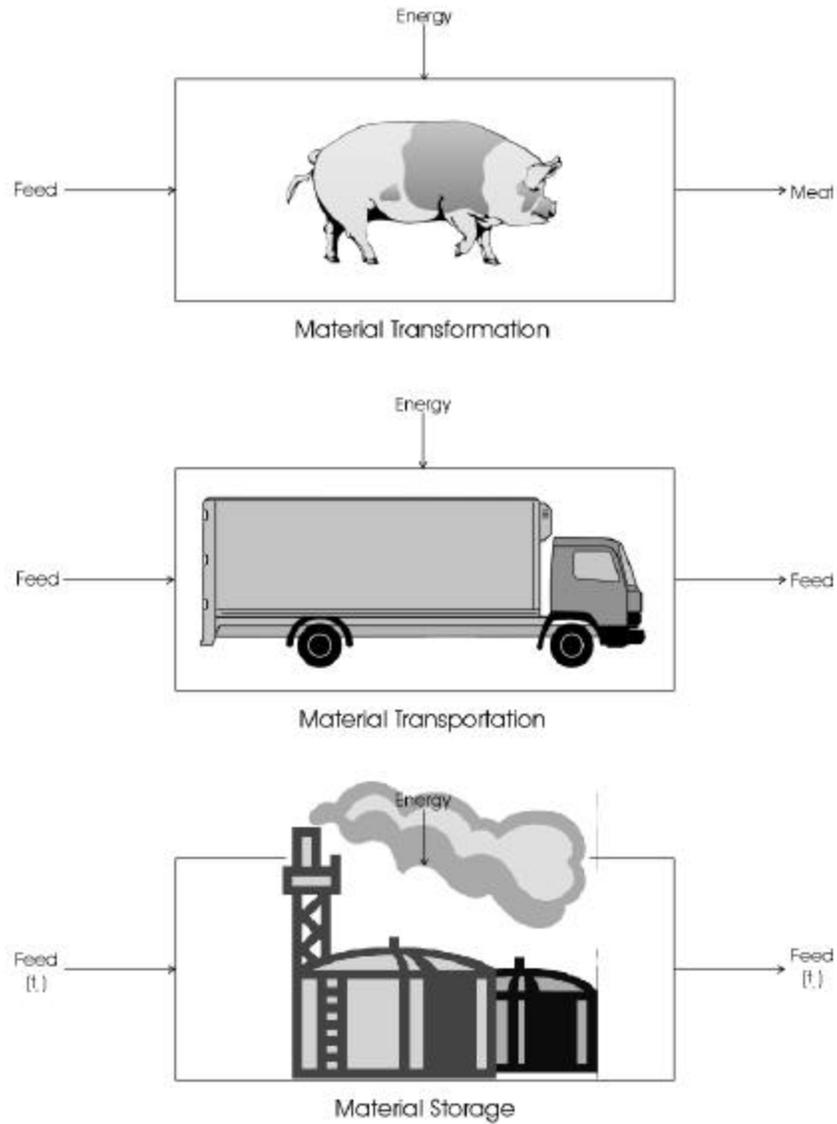


Figure 2.2. Schematic representation of transformation, transportation, and storage processes.

#### Network

The system components are united at finite number of interfaces by means of common sub-components.

The joining of two components is effected by a sub-component which they have in common. This is the basis upon which all biological system structures are formed. The plants and animals in a swine production farm may interface at the

biomass-feed conversion subsystem. Grain storage and product marketing may interface at the grain packaging subsystem. Food production and food distribution systems may interface at the food packaging subsystem.

Changes in the system are transmitted through the interconnections.

The phosphorus biochemical cycle illustrates the connectedness of systems and shows how one activity affects the other (Fig. 2.3). For example, phosphate mining is connected to corn production system through the flow of fertilizers; a corn production system is connected to a milk production system through the flow of animal feeds.

The network in a biological system exhibits transitivity.

If subsystem 1 is connected to subsystem 2 and subsystem 2 is connected to subsystem 3, then subsystem 1 is connected to subsystem 3. For example, if phosphate mining in California supplies raw materials to the fertilizer production in Michigan and the fertilizer production contributes to the lake pollution in the Saginaw Bay, then the phosphate mining in California contributes to the lake pollution in the Saginaw Bay (Fig. 2.3).

The network in a biological system is asymmetrical.

The roles of the components in the network are not interchangeable. For example, the soil system holds the plants, not the other way around; the harvester removes the crop, not vice versa; the animals feed on the crops, not the other way (I hope so!). The relation of the soil to the plant is not the same as the relation of the plant to the soil; the relation of the harvester to the crop is not the same as the relation of the crop to the harvester.

The network in a biological system demonstrates seriality.

If the components in the system are transitive and asymmetrical, then they are serially connected. For example, the pollution of Saginaw Bay by phosphates and the application of phosphates in crop production and the production of feed for animal production has a serial relationship.

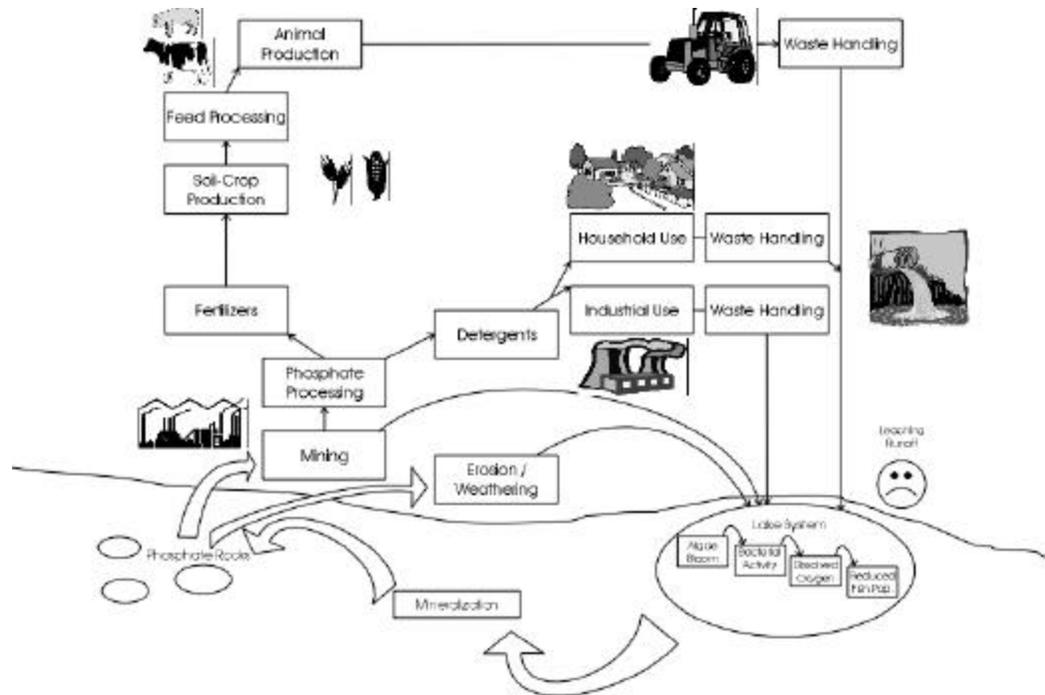


Figure 2.3. A schematic diagram of a phosphorus biochemical cycle.

The network in a biological system exhibits interdependence.

The existence of one component in the system is conditioned by some other components. For example, the quality of milking cows depends upon the quality of feeds produced in a crop-dairy system. In another examples, a series of connected lakes will exhibit interdependence.

Wholeness

A biological system exhibits coherence.

For example, a change in the feed ration of the dairy herds in a crop-dairy system may affect the milk production component of the system. A change in the marketing structure in the agroeco-regional system may alter the transportation of products from the farms. A decrease in irrigation supply may reduce the yield or the grain quality of crops.

At all levels of hierarchy, the successful and efficient operation of the whole is the main objective of the biological system.

The individual components may not be operating in an optimum fashion at a particular time, but each action is compatible with the overall system requirements for the interest of the whole biological system. For example, an animal is made up of diverse biological organs, such as the mouth, stomach, legs, body, and head. Their movements are coordinated by some biological mechanisms to achieve survival. Take the case of a cereal plant, such rice, maize, or wheat. During its growth, the leaves maximize their growth (in terms of height and weight) only at the earlier season, and then minimize their growth in favor of the panicle at the later period of plant growth. The same is true of the stems and the roots. When grain filling sets in, assimilates from the leaves and stems are sometimes translocated to the grain. At the end of the season, the whole effort of the cereal plant during the growth process is concentrated towards the production of grain.

If a component is not connected, it is not part of the system.

For example, agricultural trade agreement among some countries in Southeast Asia provides that certain products of one country flows into the market of the other countries. The countries involved in the agreement have formed a trade (system) network. If one country ignores the agreement, then that country has failed to be part of the system. As illustrated in Fig. 2.4, Production Unit 1 is part of the cooperative system while it fails to become part of the system in B. From a global perspective, the north and south hemispheres are connected through the flow of materials in the atmosphere.

The properties and characteristics of a biological system involve not only the joining of the properties and characteristics of the individual components but also the nature of their interconnections.

For example, the properties of the farm-cooperative system are different from the farm-market system partly due to their interconnections, as illustrated in Fig. 2.4. The formation of a regional food production system is more than the aggregation of the production, processing, and transport systems.

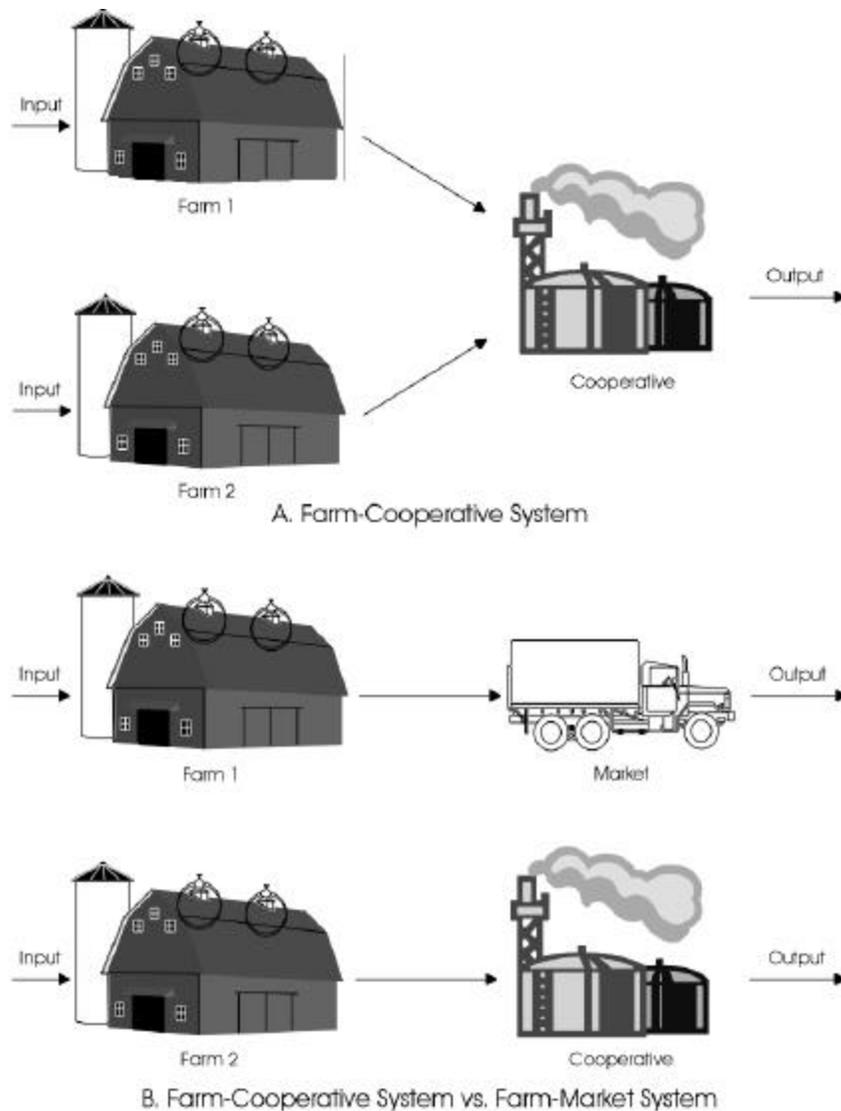


Figure 2.4. In diagram A, Farm 1 and Farm 2 are connected through the cooperative system. In diagram B, Farm 1 and Farm 2 are not connected.

The interconnected biological system may possess certain characteristics unique to itself, referred to as emerging properties, which may not be a property of any of the components.

For example, the linking of plants, soils, irrigation, and fertilizer application comprise a crop production system. Each of the components by themselves do not make a production system. The linking of farms, storage, processing, and

marketing results in a regional economic system, whereas each of the components cannot accomplish the objective of an economic market condition.

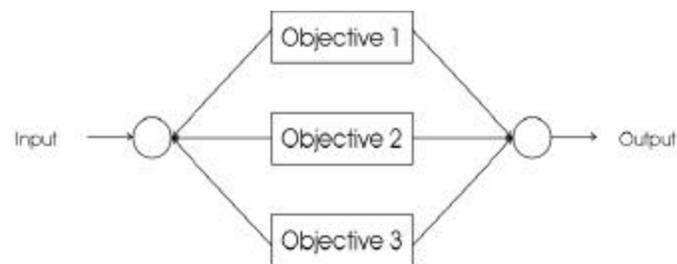
The operation of a biological system depends not only on the operation of the individual components but also on the nature of their interconnection.

Two biological systems with similar components and technical properties behave differently if their interconnections are different. For example, a crop-swine operation whose crop produce is partly sold to the market manifests a different economic behavior than when all the crops are used in the farm as feed to the swine. In a regional system, the economic condition of a system where the farms are free to sell their produce to the outside market will be different than when the farms are required to sell their produce to the cooperative. Figure 2.4 shows that in diagram A, Farm 1 is connected to Farm 2 through their joint interconnection to the Cooperative. In diagram B, Farm 1 is totally disconnected from Farm 2.

Purpose

A biological system has one or more objectives which may be required simultaneously or at different points in time.

A crop producer may have several objectives, such as to maximize profit, minimize risk, minimize nitrate leaching, minimize runoff, minimize pesticide residues, maximize output, and so on. These objectives may need to be addressed simultaneously (parallel) or in sequence (cascade), depending on the priorities of the producer, as illustrated in Fig. 2.5.



A. Parallel



B. Cascade

Figure 2.5. Diagram A illustrates a parallel system; diagram B illustrates a cascade system.

The objectives of a biological system may vary with spatial orientation.

A crop producer in Michigan may have different objectives than someone in Iowa or Illinois due to environmental factors, political climate, market situation, cultural factors, support services, etc. Their spatial orientation will contribute to their variation (Fig. 2.6).

The objectives of a biological system may change with organizational hierarchy.

A corn production system may have as its objective the maximum production of grain yield. A farm-level system may adopt profit as its major objective. A regional-level system may be guided by environmental and sustainability issues for its goals. An economic system at the national level may be guided by political objectives.



Figure 2.6. A food producer in Michigan will have different objectives and constraints than someone from another state due to spatial variability.

The objective of the biological system defines the structuring of the system components so that they contribute to the optimum performance of the whole biological system.

A crop producer sets his/her objectives first before designing the crop rotation in a cropping sequence. The rotation and management strategies may vary depending on whether he/she prefers profit over environmental preservation, or whether he/she prefers corn over wheat or soybean.

A biological system follows the principle of equifinality.

A system can reach the same final state through different initial conditions and by a variety of ways (von Bertalanffy, 1969). For example, environmental sustainability can be achieved through many different policy initiatives and by many different methods.

The choice of the system components may be influenced by the availability of inputs, market information, facilities, and other factors that may represent the “state of the art.”

A producer’s choice to grow corn over wheat is partly dictated by his/her knowledge of the crop or market demand for the crop. His/her choice for dairy production may be dictated by the facilities he/she can afford to secure or technical skills required to operate.

Open System

A biological system exchanges matter or energy with its environment.

The plants exchange energy with the environment: the environment provides carbon dioxide to the plants; the plants release oxygen to the environment. The plants use up the nutrients from the soil and returns organic matter to the soil. The feed supply, market prices, input supplies, drugs, and services influence the structuring and management of a swine production system. The swine-production system, in turn, supplies the meat and meat products and by-products, such as manure and odor.

A biological system transforms materials from one form to another.

Photosynthates are converted into biomass in a plant system; feed is transformed into milk in a dairy cow.

The components of a biological system are in continuous building up and breaking down.

The cell needs continuous supply of energy to maintain and rebuild its parts; the plant organs grow and die; the crop-dairy system needs materials, labor and energy to replenish those parts which deteriorate or die.

A biological system attains a steady state, not in true equilibrium.

In steady state, there is a continuous inflow of energy from the external environment and a continuous outflow of products of the system into the environment, but the character of the system and the relation of the components are the same (Katz and Kahn, 1969). For example, the catabolic and anabolic processes of tissue breakdown and restoration within the body preserve a steady state so that the body is not the identical organism it was but a highly similar organism (Katz and Kahn, 1969). This property is also demonstrated in the homeostatic processes of temperature regulation in the body where the body temperature remains the same even if external temperature varies.

The products exported by a biological system into the environment provide the sources of materials and energy for the repetition of the cycle of biological system activities.

The carbon dioxide exhaled by humans and animals into the atmosphere is needed by plants; plants export oxygen into the air which is needed by humans and animals.

A biosystem acquires negative entropy.

The entropic process is a universal law of nature in which all forms of organization move toward disorder or death (Katz and Kahn, 1969). A biological system survives by maximizing the ratio of imported to expended energy, thus acquiring negative entropy. For example, microorganisms, fish, plants, animals, and other living organisms replenish energy continuously in order to survive. A crop-dairy system will seek to improve the survival and profitability of the operation.

#### Feedback

A biological system has internal feedback mechanism for survival.

Feedback enables the system to correct its deviations from the desired course by re-introducing part of the output or state behavior at the input in order to affect the succeeding output. For example, surviving plants under drought conditions have feedback mechanisms to maximize available moisture by controlling the stomatal opening. In most plants, stomates are open during daylight and close during the night. Stomates are open to allow for the assimilation of carbon dioxide needed for photosynthesis. However, if there is low moisture, the stomates close, providing a protective mechanism during drought periods. When carbon dioxide is low in the intercellular spaces and guard cells, the stomates open to allow carbon dioxide to diffuse in. This is necessary feedback for photosynthesis to complete. Certain succulent plants conditioned to hot, dry conditions, such as a cactus, open their stomates at night, fix carbon dioxide into organic acids in the dark, and close their stomates during the day so as to conserve water during the hottest part of the day.

A biological system possesses the ability to react and adapt to the environment for the continued operation of the whole system.

A prolonged environmental stress may lead to genetic changes or natural restructuring of the system. For example, plants under drought conditions behave differently from those under favorable conditions. Arid places have different vegetation than the fertile lands, or tropical plants grow in the tropics while temperate plants grow in the temperate regions. Some fruit trees avoid freezing during low temperatures by forming ice crystals in the spaces between cells which protects the tissues from low temperature damage. Adaptation can also be observed in insects becoming immune to insecticides or fungi becoming immune to fungicides.

#### Stability

The instability created by perturbations on the biological system variables is usually temporary.

The perturbed variables may eventually return to their defined, stable limits when corrective measures are applied at an appropriate period of time. For example, the physiological processes of a plant may be affected in the presence of drought conditions. It has been observed that during drought period, the plants allocate more carbohydrates for growth to the roots, causing the roots to extend

deeper into the soil “in search of alternate water sources.” During this time, the shoot growth is slowed down in favor of the roots. However, when water is eventually supplied, the plant may be able to recover and the physiological processes may return to normal as if the plant has not experienced any stress at all. Another example is the stability mechanism in the cow. During extremely hot weather, the cows respond to the environment by increasing their respiration which may result in the reduction of milk. However, milk production returns to normal when the hot weather is over. In cases where the perturbation is prolonged and corrective measures are not applied, the biological system may be altered irreversibly. Desertification is one irreversible damage that can happen with massive deforestation.

#### Growth

The living components of a biological system has the ability to absorb substances, grow, and reproduce.

A rice seed breaks and grows to a full plant bearing more rice seeds. This growth process is made possible by the plant’s ability to absorb the required amount of nutrients, water, and sunlight.

A biological system will tend to move in the direction of differentiation and elaboration.

For example, a food production system will tend to move in the direction of product specialization, product diversification, and progressive mechanization.

#### Stochasticity

Some system variables of a biological system are affected by factors that are random in nature or events that occur by “chance.”

In stochastic events, probability theory is required to make predictions about the output. For example, the plant system is highly affected by the stochastic nature of weather variables (rainfall, solar radiation, air temperature, wind velocity, humidity, etc.) and the farm enterprise is affected by the stochastic market conditions. The uniformity of water distribution in an irrigation system is influenced by the random behavior of the wind speed and direction. Rain is good for the crop but if it comes at the time when the field is still being prepared for planting, it may not be a welcome event. Too much water can also lead to waterlogging which affects the growth of plants. Frost may come at the time when plants are most sensitive to temperature stress. Hail, storm, typhoons, and cyclones are oftentimes damaging to agricultural production. Weather may affect the market situation due to fluctuations in the prices of commodities. When

production is bad in one area due to the weather, some commodity prices change and another area may need to supply the market demand for the commodity.

## Case Studies

1. The article entitled “Persistence of the DDT Pesticide in the Yakima River Basin Washington” (Rinella et al., 1993) provides a medium to understand the systemic properties of a biological system while at the same time illustrate how the natural environment, a precious natural resource, can be so damaged by a seemingly “friendly” technology—the pesticides.
2. USGS Circular 1225 entitled "The Quality of Our Nation's Waters: Nutrients and Pesticides" shows how economic development is tightly connected to environmental quality. It presents issues where nitrogen and phosphorus, so essential for healthy plant and animal populations, can become sources of discomfort and threat to people and the environment. Reference: <http://water.usgs.gov/pubs/circ/circ1225/>
3. USGS Circular 1153 entitled "A Strategy For Assessing Potential Future Changes in Climate, Hydrology, and Vegetation in the Western United States" demonstrates the interconnectedness of weather, water, and plants. Reference: [http://greenwood.cr.usgs.gov/pub/circulars/c1153/c1153\\_1.htm](http://greenwood.cr.usgs.gov/pub/circulars/c1153/c1153_1.htm)

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## Systems Methodologies

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### PRINCIPLES

For some class of biological system problems, there is a present state and a desired state, and there are alternative ways of getting from one state to the other.

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The basic philosophy of the systems methodology, as applied here, is that for some class of biological system problems, there is a present state,  $S_0$ , at time  $t_0$ , and a desired state,  $S_1$ , at time  $t_1$ , and there are alternative ways of getting from  $S_0$  to  $S_1$ . That is,

Present state  $S_0(t_0) \rightarrow$  Desired future state  $S_1(t_1)$

For example, there are several ways of going from a person's house to the city. The options include walking, driving a car, riding a bus, taking the train, flying in an airplane, taking the boat, etc. The systems methodology is both science and art. Many times, problems are defined and solutions are discovered when the mind is relaxed and the environment pleasant. So, before proceeding to the systems methodology itself, here are some exercises to help stretch our imagination in finding alternative (creative) ways of solving problems.

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### TOPICS

Creative Thinking  
 General Systems  
 Methodology  
 Life Cycle  
 Assessment  
 Biological Modeling  
 Data Analysis

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### 3.1 Creative Thinking

#### Exercise 31. Remove Mental Locks

“Discovery consists of looking at the same thing as everyone else and thinking something different.” (Nobel prize winning physician Albert Szent-Gyorgyi, as quoted from von Oech, 1990, p. 7)

**Problem:** Here is a Roman numeral seven (von Oech, 1990, pp. 8-9): VII. By adding only a single line, turn it into an 8. Here is a Roman numeral nine: IX. By adding only a single line, turn it into a 6. By adding another line, find another way to turn it into a 6.

#### Exercise 32. Think Differently

“Nothing is more dangerous than an idea when it's the only one you have.” (French philosopher Emile Chartier, as quoted from von Oech, 1990, p. 26).

**Problem:** Put a sheet of newspaper on the floor (von Oech, 1990, p. 8). Find a way to let two people stand face to face on it, but not able to touch one another. No cutting or tearing of the paper nor tying them up or preventing them from moving is allowed.

**Exercise 3.3 Look for the Second Right Answer**

“The best way to get a good idea is to get a lot of ideas.” (Nobel prize winning chemist Linus Pauling, as quoted from von Oech, 1990, p. 26).

**Problem:** Four shapes are shown in Fig. 3.1. Select the shape that is most different from the others (von Oech, 1990, p. 22). Explain your choice.

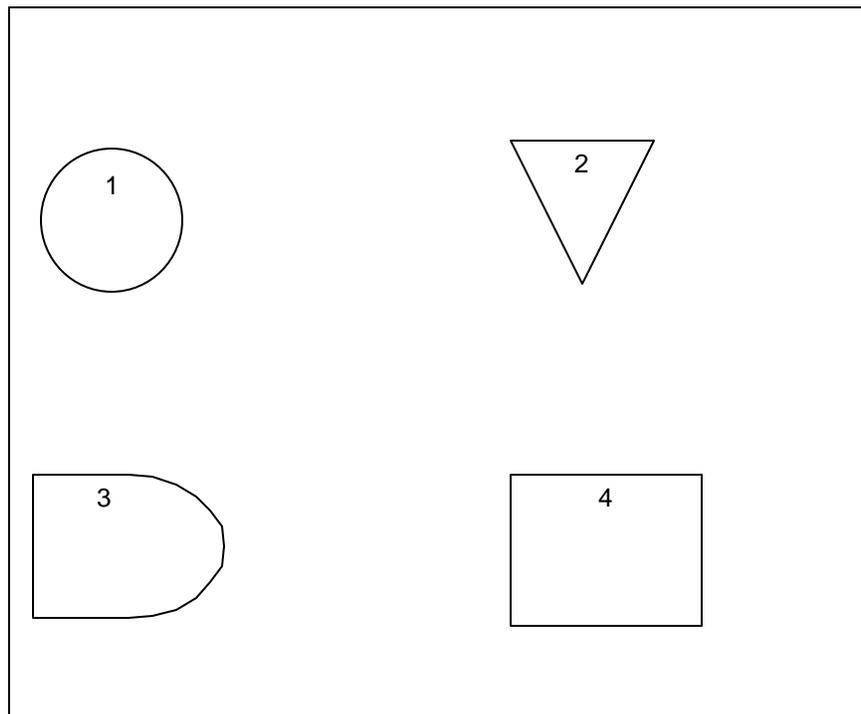


Figure 3.1. Which shape is the most different?

### 3.2 General Systems Methodology

The systems methodology is not designed to make decisions but to enable the decision-maker to ask the right questions. It is a procedure that systematically evaluates and defines the problem from a multidisciplinary point of view, identifies the true needs and ultimate objectives, constructs alternative solutions, and evaluates these options according to the technological effectiveness, cost, environmental risks, and other evaluation criteria of interest.

The systems methodology is initiated when there is a problem, or when a solution is vague and fuzzy. For example, an economic enterprise or an agricultural economy may desire to increase profit, expand production facilities and products, become competitive in the local or world market, reduce operating costs, adopt to new technological innovations, respond to policies or regulations, adjust to government subsidies, concern for the preservation of the environment, and cater to the social milieu. Sometimes, the systems methodology is classified into “hard” and “soft” (Checkland, 1981). “Hard” systems methodology is considered goal-oriented and suited well to problems where the desirable goals are clear and well defined. Mathematical modeling is often the approach used to find the optimum solution. On the other hand, “soft” systems methodology is considered a systems-based means of *structuring a debate*, rather than as a recipe for guaranteed efficient achievement (Checkland, 1981).

The socio-political component of a biological system, such as politics, customs, preferences, personal biases, social values, moral values, aesthetic notions, etc. introduces fuzziness to any biological system problem. Identifying the true problem and setting the objectives could lead to “debate” where “soft” systems methodology could be helpful. However, the problem may include natural or biological components, which need to be “engineered.” In this case, mathematical modeling may become a necessary process in finding the best solution. Therefore, in biological systems problems, there is a place for a combination of both “hard” and “soft” systems methodologies. The systems methodology presented in this book is an attempt to provide a structure where both methodologies are blended to develop an approach that is most appropriate and effective for any biological system problem.

Following the general systems engineering paradigm, the systems methodology presented here is composed of five major stages, namely: problem definition, goal setting, systems synthesis, systems evaluation, and system selection, as illustrated in Fig. 3.2. These five stages operate in an iterative manner, that is, the problem-solving process can be repeated many times until a satisfactory result is generated.

Problem definition and goal setting concern with what the job is; systems synthesis and evaluation concern with how to do the job. Problem definition and system selection operate in the “Existent” world (“what is”) while goal setting, systems synthesis, and systems evaluation operate in the “Systems” world (“what should be”). A structured debate could occur during the problem definition and goal setting stages. The major activities in each stage of the methodology are presented in a tabular form below.

### **Problem Definition**

Problem definition is the stage of need awareness and expression. The problem is recognized when there is an indeterminate situation, such as a vague objective, fuzzy approach to achieve a goal, and feelings of uncertainty. According to Hall (1962, p.93) “the confusion is a function of both the situation and the

inquirer.” The objective of this stage is to display and express the problematic situation so that the true need can be isolated from the perceived need and the appropriate solution or solutions can be identified.

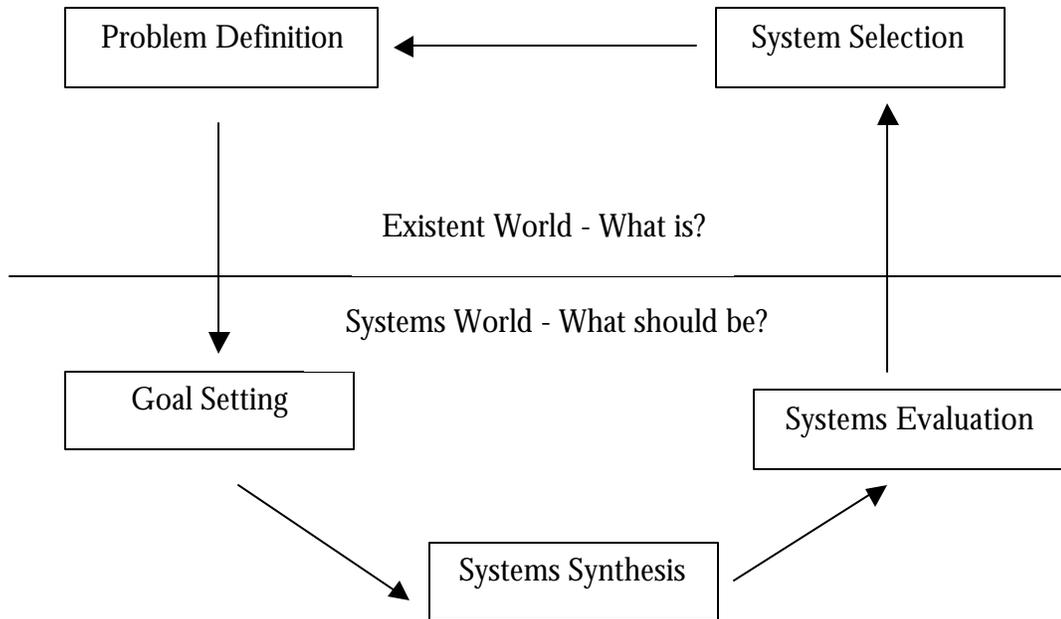


Figure 3.2. Framework of the general systems methodology.

Stage	Activities
Problem definition	Understand the real-world problem; state the problem.
Goal setting	State the objectives; specify requirements; define performance; define cost measures.
Systems synthesis	Explore alternatives; integrate systems; design systems.
Systems analysis	For each alternative system:  Decompose functions; design interfaces; modify system; evaluate risk of system; establish total quality management and project management for the system.
System selection	Select best solution; documentation.

Problem definition is an art in the sense that there are no straightforward guidelines that would guarantee to generate a succinctly defined problem. However, the following steps are outlined to help put the problem in perspective.

- Gather as many perceptions of the problem as possible from a wide range of people with roles in the problem situation in order to understand the different perspectives, while trying not to impose your own observation. Be sure that all parties involved in the problem are heard. Collect such information as control hierarchy, personal desires, economic considerations, personal preferences, etc.
- Identify the true need as against the perceived need. The true need usually defines the general, long-range problem.
- If multiple needs are identified, select one that would be studied further. This is the problem you want to pursue.
- Collect as many facts as possible related to the identified need. These facts will help define the root of the problem and will play a major role in developing and selecting the solutions. Specifically:
  - Define the level of functional hierarchy.
  - List, define, and describe all possible controllable and exogenous inputs, in terms of material flux, energy costs, and information (technology).
  - Define the desired products and undesired by-products of the biological system.
  - Define the processing components (processes or operations) associated with the inputs and outputs.
  - Define the boundary conditions between the biological system and the environment.
  - Define the constraints of the biological system, that is, describe all other boundary conditions that limit the area of feasible, permissible, or acceptable solutions.
- Give the biological system a name.
- Identify the time frame of analysis.

### **Goal Setting**

Goal setting is the stage where objectives are defined and stated in quantitative and measurable terms. These objectives may reflect profit (long or short-terms), market supply, cost, quality, efficiency, reliability, compatibility, adaptability, permanence, simplicity or elegance, safety, or time (Hall, 1962, pp. 105-107). The objectives guide the search for alternative solutions, imply the type of analysis required of the alternative approaches, and provide the criteria for selecting the optimum system (Hall, 1962, p.9).

Some guidelines are presented for setting objectives (these are patterned after general systems methodologies):

- Strive for a free flow of expressions and accept all reasonable statements.
- Write down objectives and check for definiteness of purpose.
- Check for consistency and logical coherence of objectives at all functional levels.
- Check for completeness (or exhaustiveness) of objectives by looking out for measurable statements, such as profit, market, cost, quality, performance, competition, compatibility, adaptability, longevity, simplicity, safety, legal, social, and environmental constraints, owners, implementors, beneficiaries, victims, etc. However, be aware that it is not possible to ensure completeness during the initial stages of problem-solving. Completeness may be improved as the methodology iterates.
- Incorporate risk and uncertainty in the objective statements.
- Identify the ultimate objectives. If agreement is lacking, appeal to authority, if necessary.
- Give each objective a system of measurement. These measurements are important during the implementation phase to determine whether the objectives are attained or not.
- Check to see that each objective is physically, financially, economically, technically, environmentally, legally, psychologically, and socially feasible. If limiting factors are known, state them clearly but concisely.
- Resolve value conflicts by identifying logical, factual, and value questions from among interested parties. Use tentativeness: the possibility of modifying the objectives at a later date. Avoid dogmatic attitudes, dictatorial approaches, and premature voting.

### **Systems Synthesis**

Systems synthesis is the stage of generating alternative solutions. During this ideation stage, components are put together to form alternative systems that can satisfy the objectives. Systems synthesis is a very important part of problem-solving because the subsequent stages of systems evaluation and system selection are based from the pool of ideas generated during this stage. Constructing biological system models is a major feature of this stage. The assembling of system components to form an alternative biological system is done through mathematical modeling.

The general strategies of systems synthesis are:

- Generate alternative systems. Gather all known alternatives from literatures, scientists, policy-makers, experienced people, and other sources. Do not reject any alternative system at this stage. Encourage free expression of ideas, both current and old-fashioned. However, if the problem has never been done before, be creative and seek the help of experts in the relevant fields.
- Use networking techniques, such as block diagrams and flow graphs, to set up component models and their interconnections.
- Delineate subsystems or processing components and give each a name. Identify inputs and outputs in terms of material flow rates, energy cost functions, and technology for each subsystem or processing component. Define the time-frame of analysis, the level of functional hierarchy, the process associated with the inputs and outputs, the desired product and undesired by-products, and the boundary between the processing component and its environment.
- Define the biological system structure through the component interconnections. Determine logical and time sequence.

### **Systems Evaluation**

Systems evaluation is the stage where alternative systems generated during the systems synthesis stage are analyzed and consequences are derived with respect to the stated objectives, assumptions, and boundary conditions in order to prepare for the system selection stage. The objective of systems evaluation is to anticipate relevant consequences that might render the system useless during the implementation phase. This stage helps to identify the appropriate system with minimum risk of failure. Therefore, uncertainty of consequences must be accounted for by making probability statements about the outcome through empirical data or intuitive perception.

There are five skills that are useful to a biological system engineer in the same way that they are useful to any systems engineer (Hall, 1962 pp.114-115), namely:

- Able to select the right tool for a given analysis.
- Knows when to slow down and acquire greater facility with a given tool.
- Able to speak the language of experts when consultation is required.
- Knows the limits of usefulness of a given tool.
- Knows when to stop analyzing and make a decision.

### **System Selection**

System selection is the stage where the “best” alternative system is selected - “best” in the light of the objectives, assumptions, boundary conditions, and the relevant consequences. The selection process may not be straightforward, considering the uncertainty and interdependent behavior of the components in a biological system. In such cases, the decision-maker may have to rely on authority, experience, intuition, or some personal criterion.

When the biological system problem is fuzzy due to the socio-political components, the system selection stage may involve a process of choosing the “best” solution from among a set of optimum solutions, a process called multicriteria or multi-objective optimization. The definition of “best” is subject to the cultural, moral, political, or personal preference of the decision-maker.

### **Documentation**

Since problem-solving involves a thorough search of alternative systems to satisfy a need, the ideas and alternative systems that have not been accepted are still important and may be of use in the future. There are four ways to handle rejected alternatives, as adapted from Hall (1962 p. 122). First, it should be recognized that the rejected biological system models are unwanted only for this case, regarding the rejection as tentative. New events may lead to a use of these rejected alternatives. Second, subsystems of the rejected biological system may be useful to improve on the selected system. Third, rejected alternative biological system models represent the results of creative thinking therefore they should be preserved with great care. Fourth, the reasons for rejecting an alternative biological system must be made clear in an unbiased manner. Hall further suggested to include a suitable section in the final report that discusses the reasons for rejecting the alternative system.

### 3.3 Streamlined Life Cycle Assessment (SLCA) for Comparing Two Products

Life-cycle assessment or LCA is a way of evaluating the total environmental impact of a product through every stages of its life -- from obtaining raw materials, such as mining or logging, all the way to factory production, product distribution, retail, use, and disposal, which may include incineration, landfill, or recycling. Streamlined life cycle assessment or SLCA presented here is a simplified version of the LCA methodology. The SLCA methodology in this section is a synthesis of SLCA materials and the author's experience from classroom instruction to undergraduate engineering students. For purposes of instruction, the SLCA presented in this book is specific to the comparison of products having similar utility. It is presented as a process to evaluate the environmental burden, health impact, and resource consumption associated with two comparable products by identifying the material and energy usage and environmental releases, assess the impact of those material and energy uses and releases on the environment, and evaluate opportunities to effect environmental improvements.

#### **Definition**

The Society of Environmental Toxicology and Chemistry (Todd and Curran, 1999) defines LCA as a process to evaluate the environmental burdens associated with a product system, or activity by identifying and quantitatively describing the energy and materials used, and wastes released to the environment, and to assess the impacts of those energy and material uses and releases to the environment. The assessment includes the entire life cycle of the product or activity, encompassing extracting and processing raw materials; manufacturing; distribution; use; re-use; maintenance; recycling and final disposal; and all transportation involved. LCA addresses environmental impacts of the system under study in the areas of ecological systems, human health and resource depletion. It does not address economic or social effects.

LCA studies the environmental aspects and potential impacts throughout a product's life (i.e. cradle-to-grave) from raw material acquisition through production, use and disposal. The general categories of environmental impacts needing consideration include resource use, human health, and ecological consequences.

#### **Methodological Framework**

Figure 3.3.1 illustrates the LCA framework, which consists of four phases: goal and scope definitions, inventory analysis, impact analysis, and interpretation (Graedel, 1998). The double arrows between the phases indicate the interactive and iterative nature of LCA. For example, when doing the impact analysis, it might become apparent that certain information is missing, consequently, the inventory analysis must be improved.

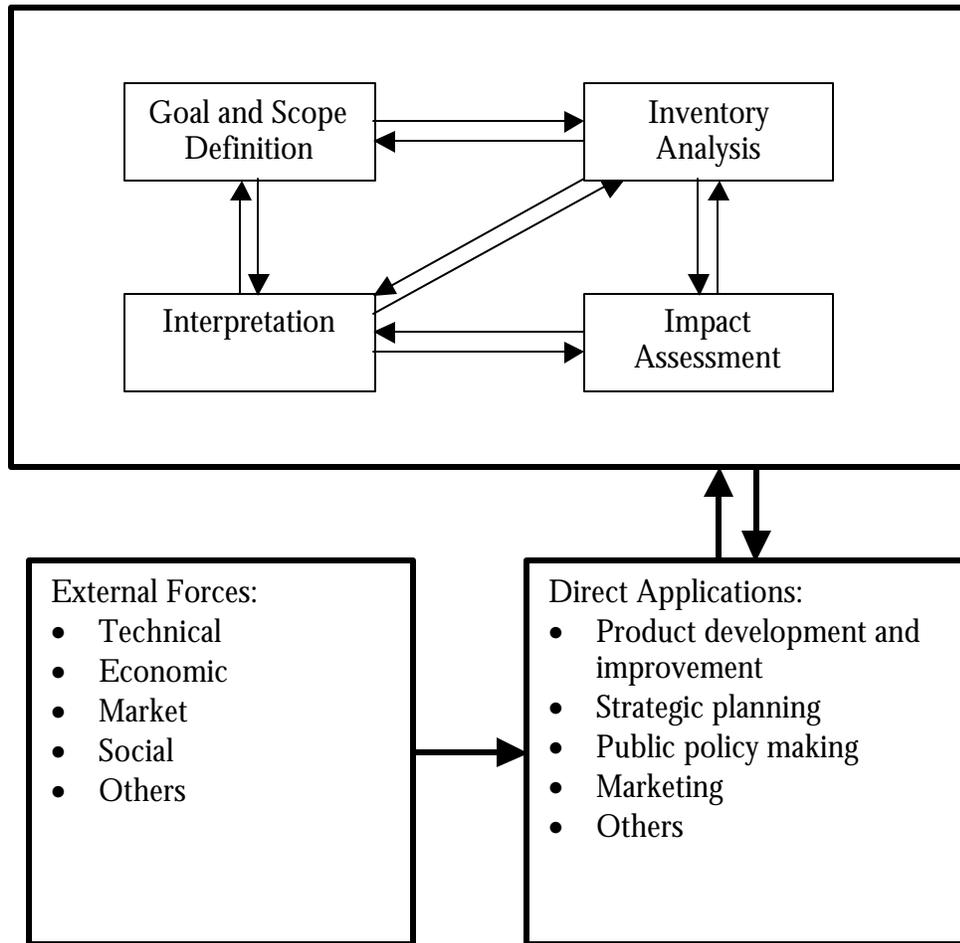


Figure 3.3.1. Methodological framework of life cycle assessment.

In order to understand SLCA, it is useful to look at the total material budget from natural resource extraction to production until its return to the reservoir of its origin. An ecological look of a material cycle is illustrated in Figure 2.1 (Chapter 2).

**Phase 1. Goal and Scope Definition**

The purpose of Goal and Scope Definition is to evaluate alternative designs in order to maintain economic feasibility while also seeking to reduce the environmental damage due to the current conventional design. It consists of goal, flowchart, scope, functional unit, system boundaries, and data quality.

**Goal**

The goal of an LCA study shall clearly state the intended application, reasons for carrying out the study and intended audience and user of results (Todd

and Curran, 1999). Examples of goal statements are: (1) to compare two or more different products fulfilling the same function with the purpose of using the information in marketing or regulating the use of the products, (2) to identify improvement possibilities in further development of existing products or in innovation and design of new products, (3) to identify areas, steps etc. in the life cycle of a product where criteria can be set up as part of eco-labeling by a regulatory board.

For example, an SLCA of alkaline and nickel batteries (DeKleine et al., 2001) may be expressed as follows: the purpose of this SLCA study is to compare the environmental impact of alkaline and rechargeable nickel alloy batteries as a portable power supply. Another example is to compare and evaluate the environmental impact of gasoline and ethanol as motor fuel (Radtke et al., 2001).

### ***Goal Formulation***

Goal formulation states the background and rationale of the goal or objective of the study. In the battery example (DeKleine et al., 2001), goal formulation was stated as follows: “As technology grows, so does the need for batteries. Although conventional alkaline batteries meet all federal guidelines for safe disposal and incineration, they are not necessarily good for the environment. They use nonrenewable resources and represent a large volume of toxic materials in the landfills. Dry cell batteries are made up of three components: the anode, cathode, and electrolyte. In an alkaline battery the anode contains high purity zinc powder, the cathode contains electrolytically produced manganese dioxide, and the electrolyte is a concentrated potassium hydroxide solution. During cell discharge, the oxygen-rich manganese dioxide is reduced and the zinc becomes oxidized, while ions are being transported through conductive alkaline electrolyte. The rechargeable nickel metal hydride is composed of the positive and negative electrodes. The positive electrode contains nickel oxyhydroxide, and the negative electrode contains a metal hydride. An aqueous potassium hydroxide solution is the primary electrolyte. Minimal electrolyte is used to facilitate oxygen diffusion to the negative electrode. During charging the process is reversed. Battery waste disposal is not regulated under RCRA (Resource Conservation and Recovery Act) and therefore it may be disposed in the trash. While new Municipal Solid Waste (MSW) landfills are designed to handle small amounts of hazardous wastes, these wastes can be better managed in a designated program for collection or recycling.”

In the fuel example (Radtke et al., 2001), the formulation was stated as follows: “Crude oil is pumped from oil fields in different areas around the earth. Extensive exploration has been required and will continue to be used to find sources of oil. These oil fields are often located in inaccessible areas far away from where the product is to be used. This has led to the development of an extensive transportation system. However, since oil is a nonrenewable resource, there is a finite supply and the economically accessible crude oil will eventually be completely consumed. Geologists have developed several reliable methods for finding oil in the earth. The methods range from using sensitive magnetic detection equipment to producing seismic events with explosives. Unfortunately, positive

identification can only be made through the drilling of a test well. Ironically, the best of these oil fields are located in inaccessible, ecologically sensitive areas. This means that they are far from where the final products will be consumed. This inaccessibility requires extensive transportation efforts and generates substantial costs. During this transportation process, some crude oil will inevitably be released into the environment. These losses account for an estimated 35% of all of the oil present in the oceans. Upon the completion of the transportation process, the crude oil arrives at the refinery. On the other hand, ethanol can be produced from most commercially grown crops. Crops containing high levels of sugar and starch, such as sugar cane, potatoes, sugar beets, and corn, are most commonly used. For the purpose of illustration, only corn will be examined. Corn production is relatively simple compared to other ethanol producing biomass. Equipment requirements are minimal and new varieties of corn have decreased the need for disease and pesticide control. Corn needs water, sun, nitrogen, and fertilizers to grow. Nitrogen is required for corn production. This can be met with the application of any nitrogen product such as urea, ammonium nitrate, or manure. Other required nutrients include phosphorus and potassium. Corn is harvested and transported to an elevator or distillery.”

### **Flowchart**

Flowcharting is important in understanding the process under consideration. Making the process flow explicitly aids several decisions of the overall analysis<sup>1</sup>: deciding the scope of the analysis to be performed, determining what data needs to be collected, identifying at what point in the process environmental loading originates, formulation of process alternatives, and judgment regarding streamlining. After analyzing the process flowchart, it may become evident that several different approaches could be taken to reduce the amount of emission to the environment. If so much is put in landfill, an option might be to incinerate and recycle more of the total. However, incineration leads to higher emissions into the atmosphere and recycling has prohibitively high cost. The best option for reduction might be to target the initial source.

Back to the fuel example, the SLCA flowcharts of the gasoline and ethanol production processes are presented in Figures 3.3.2 and 3.3.3, respectively (Radtke et al., 2001).

### **Scope**

Scope defines the boundary of the assessment -- what are included in the system and what detailed assessment methods are to be used (Todd and Curran, 1999; Graedel, 1998). In defining the scope, the following items shall be considered and clearly described: the functions of the system; the functional unit; the system to be studied; the system boundaries; allocation procedures; the types of impact and the methodology of impact assessment and subsequent interpretation to be used; data requirement; assumptions; limitations; the initial data quality requirements; the

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<sup>1</sup> <http://tiger.eea.eu.int/projects/EnvMaST/lca/>

type of critical review, if any; and the type and format of the report required for the study. The scope should be sufficiently well defined to ensure that the breadth, the depth and the detail of the study are compatible and sufficient to address the stated goal.

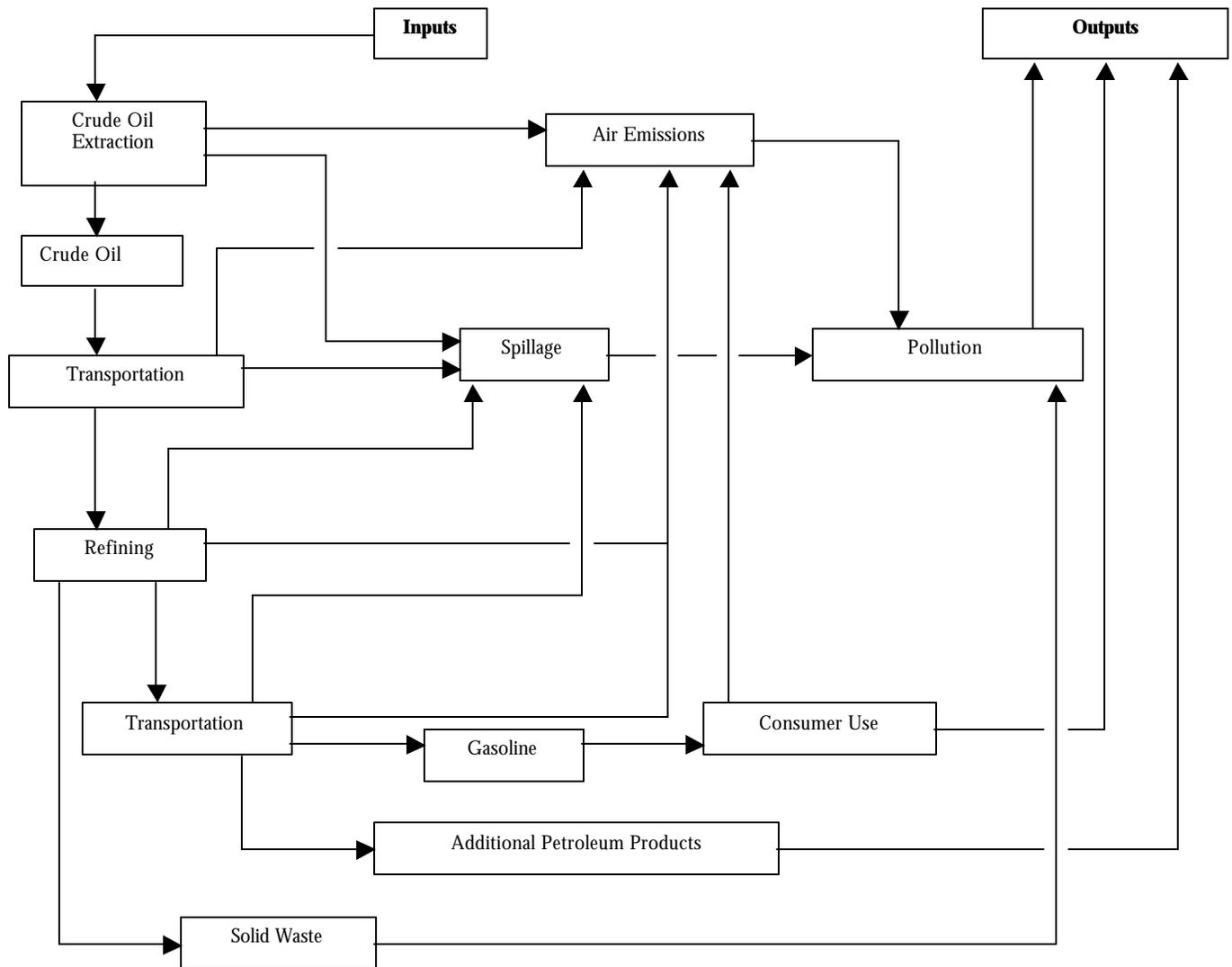


Figure 3.3.2. Schematic life cycle flowchart of gasoline production system (Radtke et al, 2001).

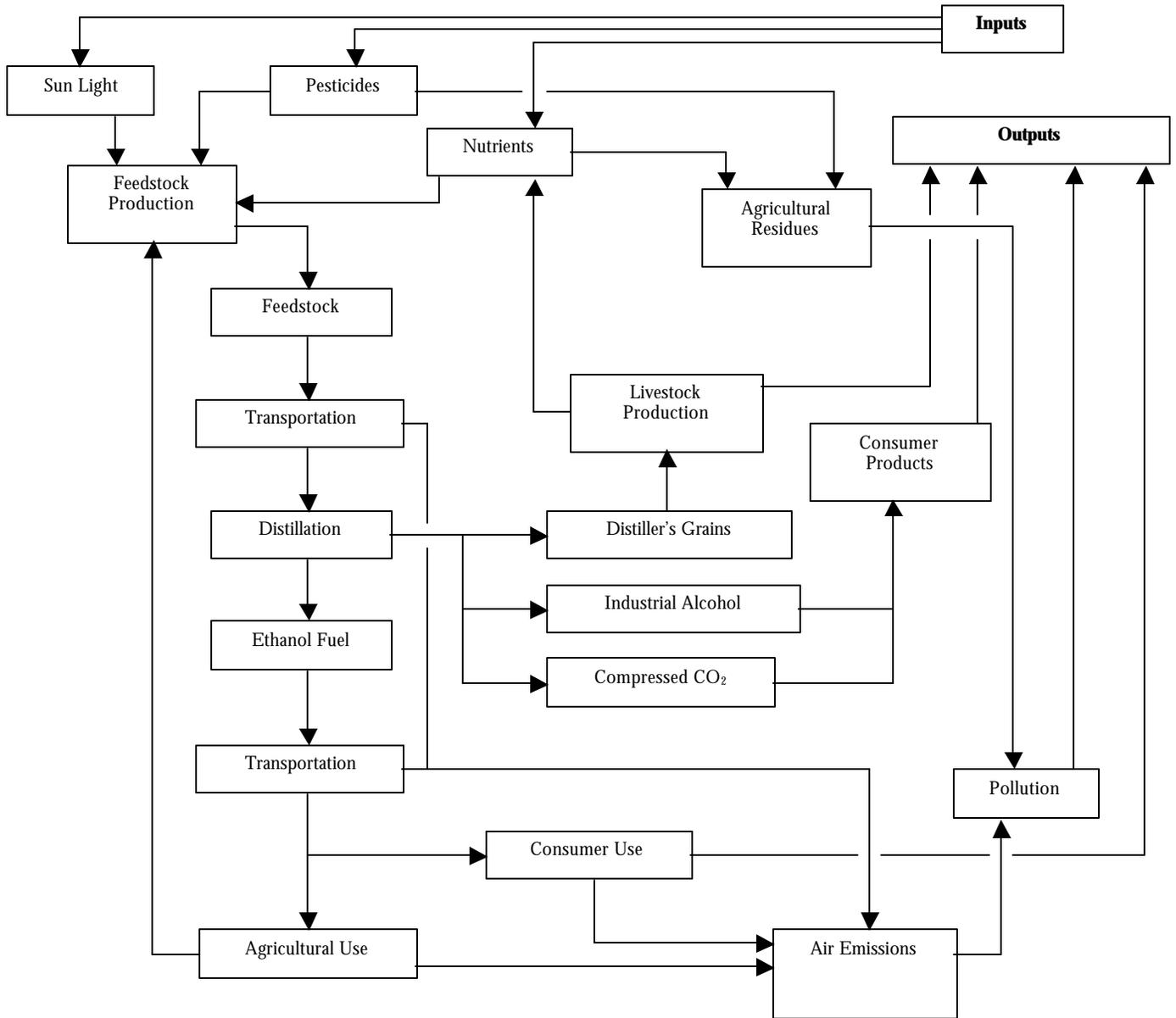


Figure 3.3.3. Schematic life cycle flowchart of the ethanol production system (Radtke et al, 2001).

In the battery example, the problem was bounded to reduce the amount of batteries released in landfills to prevent more long-term environmental damage done by the release of the toxins from batteries (DeKleine et al., 2001). The identified constraints were:

- **Overriding Constraints** - The scope of the battery case study was dictated primarily by the focus of the long-term environmental impact of the two types of batteries.
- **Constraints Due to Comparison** - Since the two products were similar in processing, the study was limited to those areas in which the two products differed. Transportation and packaging were assumed to be similar and therefore were not included in the analysis.
- **Data Constraints** – Most of the data were estimates due to lack of actual numbers.
- **Lifetime Constraints** - The shelf life of the alkaline battery was slightly higher than nickel batteries.

### ***Functional Unit***

The functional unit sets the scale for comparison of the two products (Todd and Curran, 1999). All data collected in the inventory phase will be related to the functional unit. When comparing different products fulfilling the same function, definition of the functional unit is of particular importance. One of the main purposes of a functional unit is to provide a reference to which the input and output data are normalized. A functional unit of the system must be clearly defined and measurable. The result of the measurement of the performance is the reference flow. Comparisons between systems shall be done on the basis of the same function, measured by the same functional unit in the form of equivalent reference flows. In the battery example, the functional unit was one AA size battery and 2,400 hours of use.

### ***Data Quality***

The quality of the data used in the life cycle inventory reflects the quality of the final LCA. The following data quality indicators shall be taken into consideration in a level of detail depending on goal and scope definition<sup>2</sup>:

- **Precision** - measure of the variability of the data values for each data category expressed (e.g. variance).
- **Completeness** - percentage of locations reporting primary data from the potential number in existence for each data category in a unit process.
- **Representativeness** - qualitative assessment of the degree to which the data set reflects the true population of interest (i.e. geographic and time period and technology coverage).
- **Consistency** - qualitative assessment of how uniformly the study methodology is applied to the various components of the analysis.

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<sup>2</sup> <http://tiger.eea.eu.int/projects/EnvMaST/lca/>

- Reproducibility - qualitative assessment of the extent to which information about the methodology and data values allows an independent practitioner to reproduce the results reported in the study.

There are several guidelines that can aid in formulating alternatives for redesign of a product: (1) Re-engineering the Manufacturing Process, (2) product modification, and (3) re-engineering disposal.

### **Phase 2 Inventory Analysis**

Inventory analysis is the second phase in LCA. The possible life cycle stages for consideration are: (1) Pre-manufacturing: raw material acquisition or resource extraction, component manufacture, material manufacture; (2) Manufacturing: material processing and product assembly; (3) Product delivery: filling, packaging, shipping, and distribution; (4) Customer use, reuse, and maintenance (including storage and consumption), (5) Recycling and waste management, and (6) Disposal. Inventory analysis includes data collection, validation, relating data to the specific system, allocation, and recycling (Graedel, 1998).

### **Data Collection**

The inventory analysis includes collection and treatment of data to be used in preparation of a material consumption, waste, and emission profile for all the phases in the life cycle. The data can be site specific or more general. Inventory analysis also involves calculation procedures to quantify relevant inputs and outputs of a product system. These inputs and outputs may include use of resources and releases to air, water and land associated with the system. The process of conducting an inventory analysis is iterative. For each of the life cycle stages under consideration, using the process flowchart as a guide, data must be collected about the inputs and outputs of each stage. (For more information, refer to the following website: <http://shogun.vuse.vanderbilt.edu/gge/CaseStudy/flbdata.htm>). These inputs and outputs may include materials (both raw and processed), energy, labor, products, and waste. In the event of multiple sub-processes being grouped together as a single life cycle stage, the data about each sub-process must be aggregated to produce a computed value (e.g. overall water use) for the whole stage. In addition, materials, energy usage, and waste produced from a specific sub-process might be an aggregate of not one, but several products produced at that point. In this case, the data must be handled carefully to ensure that the allocation of the proportion of each input and output stream to each co-product is correct (usually done by weight). Raw data collected must be converted, via the base unit and analysis time frame, to a "stand alone" state suitable for use in the analysis.

In the battery example, manufacturing, product delivery, and usage were ignored in the analysis. The life cycle stages being considered were: extraction methods and materials lost during the pre-manufacturing stage, the materials to be recycled, and the effect of the materials after disposal.

Allocation procedures are needed when dealing with systems involving multiple products (e.g. multiple products from petroleum refining). The materials and energy flows, as well as associated environmental releases, shall be allocated to the different products according to clearly stated procedures. The calculation of energy flow should take into account the different fuels and electricity sources used, the efficiency of conversion and distribution of energy flow, as well as the inputs and outputs associated with the generation and use of that energy flow.

### ***Validation of Data***

Data validation has to be conducted during the data collection process (Refer to <http://tiger.eea.eu.int/projects/EnvMaST/lca/>). Systematic data validation may point out areas where data quality must be improved or data must be found in similar processes or unit processes. During the process of data collection, a permanent and iterative check on data validity should be conducted. Validation may involve establishing, for example, mass balances, energy balances and/or comparative analysis of emission factors.

### ***Relating Data***

The fundamental input and output data are often delivered from industry in arbitrary units, such as emissions to the sewage system as mg metals/liter wastewater (Refer to <http://tiger.eea.eu.int/projects/EnvMaST/lca/>). The specific machine or wastewater stream is rarely connected to the production of the considered product alone but often to a number of similar products or perhaps to the whole production activity. For each unit process, an appropriate reference flow shall be determined (e.g. one kilogram of material). The quantitative input and output data of the unit process shall be calculated in relation to this reference flow. Based on the refined flow chart and systems boundary, unit processes are interconnected to allow calculations of the complete system. The calculation should result in all system input and output data being referenced to the functional unit.

### ***Allocation and Recycling***

When performing LCA of a complex system, it may not be possible to handle all the impacts inside the system boundary. This problem can be solved either by expanding the system boundary to include all the inputs and outputs, or by allocating the relevant environmental impacts to the studied system (Refer to <http://tiger.eea.eu.int/projects/EnvMaST/lca/>). The inputs and outputs of the unallocated system shall equal the sum of the corresponding inputs and outputs of the allocated system. Some inputs may be partly co-products and partly waste. In such a case, it is necessary to identify the ratio between co-products and waste since burdens are to be allocated to the co-product only. There shall be uniform application of allocation procedures to similar inputs and outputs of the systems under consideration.

### **Phase 3. Impact Analysis**

Impact analysis is a "quantitative and/or qualitative process to characterize and assess the effects of the environmental interventions identified in the inventory table".

#### ***Impact Categories***

The first step in impact analysis is the identification of stressors and impact categories (Graedel, 1998). Stressors are "any chemical, biological, or physical entity that causes adverse affects on ecological components, i.e. individuals, populations, communities, or ecosystems." The process of identifying stressors is necessary to assessing the environmental impacts (either real or potential). The potential hazards of an operation must be considered, then the investigator must decide which of these hazards are relevant to meet the Goal of the study. The potential stressors of industrial processes are: raw materials, energy use, air emissions, liquid discharge, solid wastes, radiation, and noise. These stressors could impact the following categories:

- Human Health - acute effects (accidents, explosions, fines, safety issues); chronic effects (injury, disease), work environment
- Ecological Health - structure (population, communities), function (productivity, processes), biodiversity (habitat loss, endangered species, eutrophication, photochemical oxidant formation, acidification).
- Social Welfare - economic impact, community impact, psycho-social impact.
- Resource Depletion - biotic resources (agricultural, forest, living species), flow resources (air quality, water quality, global warming, stratospheric ozone depletion), and stock resources (land, energy, and raw materials).

#### ***Valuation or Weighting***

The weighting of each stressor on the impact category for each life stage may follow integer rating (Graedel, 1998), where 0 represents highest impact (a very negative evaluation) and 4 lowest impact (an exemplary evaluation), or qualitative rating of high (H), medium (M), and low (L). Value assignment is arbitrary as an expression of personal preference and priorities. The approach is used to estimate the potential for improvement in environmental performance.

#### ***Example related to Phase 3***

In the battery example (DeKleine, 2001), the identified stressors were raw materials, liquid discharge, and solid wastes. Air Emission was ignored because the data showed that air emission in any of the stages was negligible. The impact categories selected were chronic health effects, biodiversity component of ecological health, social welfare, and resource depletion of the biotic and stock resources. The stressors and impact categories were weighted low (L), medium

(M), and high (H). Tables 3x.1 and 3x.2 show the average weighting of the stressors and impact categories by the SLCA team.

Table 3x.1. Weighting of stressors (DeKleine, et al., 2001).

<b>Weighting of Stressors</b>	
<b>Stressor</b>	Weight
Raw Materials	L
Liquid Discharge	M
Solid Wastes	H

Table 3x.2. Weighting of impact categories (DeKleine, et al., 2001).

<b>Weighting of Impact Categories</b>			
Impact Categories	Weight	Impact Categories	Weight
Human Health: Chronic Effects	H	Resource Depletion: Biotic Resources	M
Ecological Health: Biodiversity	M	Resource Depletion: Stock Resources	H
Social Welfare: Economic Impact	L		

After the impact categories were defined, the correlation between the stressors and the categories was made explicit. Also, the relevant stressors were assigned to the appropriate life cycle stages. Table 3x.3 shows the streamlined result.

Table 3x.3. Cross correlation of stressors and impact categories (DeKleine, et al., 2001).

<b>Cross Correlation of Stressors with Impact Categories</b>			
Impact Category	Stressors		
	Raw Materials	Liquid Discharge	Solid Wastes
Human Health: Chronic Effects	X	-	-
Ecological Health: Biodiversity	X	X	X
Social Welfare: Economic Impact	-	-	X
Resource Depletion: Biotic resources	X	X	X
Resource Depletion: Stock Resources	X	-	X

By comparing data on the health risks of the hazardous materials used in the life cycle stages of batteries, and applying it to the overall environmental impact of primary and secondary batteries and to the weighting of the impact categories table, impact analysis was developed for the primary batteries (Table 3x.4) and the secondary batteries (Table 3x.5).

Table 3x.4. Impact analysis for the primary batteries (DeKleine, et al., 2001).

<b>Primary Batteries</b>							
Life Cycle Stage	Stressor	Human Health	Ecological Health	Social Welfare	Resource Depletion	Row Average	Life Cycle Stage Average
Pre-Manufacturing	Raw materials (Zn, Mn)	L	M	L	L	L	L
Recycling	Solid Wastes	L	L	H	L	L	L
Disposal	Liquid discharge	H	H	L	M	H	H

Table 3x.5. Impact analysis of the secondary batteries (DeKleine, et al., 2001).

<b>Secondary Batteries</b>							
Life cycle stage	Stressor	Human Health	Ecological Health	Social Welfare	Resource Depletion	Row Average	Life Cycle Stage Average
Pre-Manufacturing	Raw materials (Ni, Cd)	M	L	L	L	L	L
Recycling	Solid Wastes	L	L	M	L	L	L
Disposal	Liquid discharge	M	M	L	M	M	M

The aggregated risk values for each stage for the two battery types appear in the Table 3x.6.

Table 3x.6. Aggregated analysis (DeKleine, et al., 2001).

<b>Overall Risk Aggregation for Primary and Secondary Batteries</b>		
	Primary Batteries	Secondary Batteries
Pre-Manufacturing	L	L
Recycling	L	L
Disposal	H	M

*Improvement Analysis.* The comparison of alternatives demonstrated the tradeoffs at various levels. The primary battery had a higher disposal risk, because there was no recycling effort of primary batteries. There were less rechargeable batteries in the landfills, so the risk was lower. The initial cost for secondary batteries was higher. Besides higher the initial cost, the rechargeable battery had the decreased lifecycle period. A comparison of the two batteries is shown in Table 3x.7.

*Conclusion.* Overall, the SLCA team concluded that the best battery to use was the secondary, nickel cadmium battery. Not only is there already a recycling program implemented, but since these are reusable, there are fewer batteries needed. Looking from a consumer's viewpoint, rechargeable batteries saves money.

Secondary batteries have less of an environmental impact, therefore causing less municipal waste, and fewer toxins are released in the environment.

Table 3x.7. Comparison of the two batteries (DeKleine, et al., 2001).

	<b>Primary Batteries</b>	<b>Secondary Batteries</b>
Disposal Risk	H	M
Lifecycle Period	60 hours	40 hours
Number of Lifecycles	1	Up to 1000
Cost for 4 AA Batteries	\$2.99	\$9.95 (plus \$19.95-charger)
Cost for 2400 hrs. of energy	\$120.00	\$30.00

#### **Phase 4 Interpretation**

Interpretation consists of the following steps: (1) Identify the significant environmental issues, (2) evaluate the methodology and results for completeness, sensitivity and consistency, and (3) check that conclusions are consistent with the requirements of the goal and scope of the study, including data quality requirements, predefined assumptions and values, and application oriented requirements.

### 3.4 Biological Modeling

A mathematical model is a representation of a subset of the true system expressed through the language of mathematics. These models consist of a collection of equations. The variables are associated with the physical entities and the mathematical relationships express the relations among the physical entities in the system. In other words, a relation between the variables in the mathematical expression is analogous to the relation between the corresponding physical entities (Eisen, 1988). Thus, a mathematical model of a biological system has features of the true system but not necessarily all the features of it. The features will depend on what the modeler decides to incorporate based on the purpose for which the model is built. However, there are several factors that influence the modeler's decision. First, what consists a true system depends upon the modeler's capacity of observation (Casti, 1989). Second, what components of the true system are to be represented in the model depends on the modeler's preference; it is physically

impossible to include all the components. Third, the model is constrained by the modeler's ability of mathematical expression. In summary, the success of the model depends upon the modeler's perception of what the true system is, a careful selection of what system components are to be modeled, and how these components are represented in mathematical forms. Given such factors, there will never be a truly satisfactory model for everyone and for all purposes.

### **Biological System Models**

The selection of components in modeling cannot be overemphasized because satisfactory results depend on whether the wrong things are incorporated or the right things are neglected (Eisen, 1988). On the other hand, incorporating too many components can lead to an unmanageable model and voluminous data requirement. It is important to strike a balance and capture the set of components which are necessary, resulting in a biological system model that describes the true system to a prescribed degree of accuracy (Casti, 1989; Eisen, 1988).

A major step in the development of a biological system model is the partitioning of the biological system into components. The hierarchy property of a biological system allows a processing component at one structural level to be decomposed into subsystems of a similar class to form a lower level structure. This means that the lower level is a refinement of the higher level structure in the hierarchy. The decomposition stops when all the mathematical functions of interest have been defined at the lowest level. The "lowest" level is an arbitrary level defined by the modeler.

Partitioning of the biological system into processing components is influenced by the level of mathematical abstraction the modeler is interested in. At a macroscopic level, a biological process, for example, can be expressed mathematically from empirical observations, leaving out many of the factors directly influencing its behavior. At a microscopic level, molecular and biochemical reactions may be part of the analytical expressions. For example, for some rice varieties, yield (which is in the form of carbohydrates) can be estimated from solar radiation (S) using a correlation equation developed from historical yield observations, such as (Oldeman et al., 1987)

$$\text{Yield} = 0.62S - 3.29 \quad (3.1)$$

On the other hand, yield estimation can be modeled from the level of photosynthesis, such as the following equation for a rice plant:

$$C = gS(1 - e^{-kL}) \quad (3.2)$$

where C represents the production of carbohydrates through photosynthesis,  $g$ , the genetic ability of the rice plant to convert sunlight into carbohydrates, S, solar radiation, k, the extinction coefficient that describes the leaf structure of the plant, and L, the leaf area index which describes the leaf surface (Alocilja and Ritchie,

1988). The same photosynthetic phenomenon can be modeled at the molecular level, such as (Yoshida, 1981):



where  $\text{CO}_2$ ,  $\text{H}_2\text{O}$ , and  $\text{O}_2$  are carbon dioxide, water, and oxygen molecules, respectively, and  $(\text{CH}_2\text{O})$  represents a carbohydrate molecule. The chemical reaction in equation (3.3) must occur in the presence of chloroplasts (cytoplasmic organelles that contain the chlorophyll) and sunlight (solar radiation). Yoshida (1981) further expressed photosynthesis in three partial photochemical reactions involving ATP (adenosine triphosphate) and NADPH (reduced nicotinamide adenine dinucleotide phosphate).

As demonstrated above, practically the same phenomenon can be modeled in many different ways at various level of abstraction. The choice on the level of abstraction and the method of modeling lies on the modeler. The basic guideline is to construct a biological system model such that the properties of the true biological system mirrored in the model can be explored to achieve the purpose of the modeling effort.

The level of mathematical abstraction may be guided by the degree of error the modeler may be willing to take. There are at least four sources of error. The first is the error due to the modeler's perception of what is reality. Each person may look at reality from various perspectives and therefore the interpretation of reality may differ from one person to another. The second source of error is generated between the observation and the mathematical translation. The mathematical expression is highly dependent on the ability of the modeler. The language of mathematics itself has limitations. The third source of error is the translation from mathematics to computer code. The computer is a digital representation of mathematical expressions. The fourth source of error is the digital computation. All these errors contribute in various degrees to the result of the mathematical model.

In choosing the level of abstraction, for example, equation (3.1) has only one parameter, the solar radiation. However, the equation is a correlation equation which is based on historical data from a specific site and specific variety. The equation may not be the same for other locations or varieties. So, while there are only two parameters required, the transferability of the model may be limited. On the other hand, equation (3.2) has four parameters. This equation reflects a finer model since it now includes the genetic character of the plant (through the parameter  $\mu$ ) and the plant structure (the parameters  $k$  and  $L$ ). The model is more refined but it requires more information. The refinement due to the increased number of parameters, however, may be minimized by the error in generating the parameters. For equation (3.3), more data are required related to light saturation, chloroplast conditions, and other parameters at the molecular level. Experiments at this level are tedious and may generate more experimental errors. The model

expressed in (3.3) provides more insight into the internal processes of the plant. This illustrates that the modeler has trade-offs on the level of abstraction.

A mathematical model may be formally defined as follows. A set of  $M$  mathematical equations is a model of a true biological system  $B$  if some of the variables in the set  $M$  correspond in a one-to-one manner with some of the features of  $B$  and at least some functions in  $B$  are analogous to some of the mathematical relationships in the set  $M$  (Eisen, 1988). The definition indicates that not all variables in the true biological system  $B$  are required to be present in the set of  $M$  mathematical equations and not all functions in  $B$  are to be represented by mathematical relationships in the set  $M$ . Thus, it is important to identify the variables and relationships in the biological system that are to be represented in the model. The variables are divided into two parts, namely:

- Those variables that affect the biological system but the model is not designed to study them.
- Those variables the biological system model is designed to study.

It is important to note that the biological system model will not be satisfactory if the necessary variables are neglected while the unnecessary variables are incorporated. Including too many factors can lead to a hopelessly complex model while including too little may result in an incomplete model. The modeler must exercise wisdom on which variables to reject without invalidating the results of the model. Although the biological system model may not represent all the aspects of the true system, it should describe the essential inputs, outputs, and internal states as well as provide an indication of environmental conditions similar to those of the true system.

### **Types of Biological system Models**

There are at least three types of biological system models, namely: (1) physical, (2) conceptual, and (3) mathematical. Physical models are represented by materials formed into representation of reality, such as for example, a model of a heart, face, body, muscles, animal, and plant. Conceptual models are represented by flowcharts, diagrams, and illustrations. Mathematical models are represented by equations.

### **Importance of a Mathematical Biological system Model**

Biological system models are important in many respects, including:

1. They are used to make simplified representations or observations which give results similar to those of the true system at considerably less cost, time, or inconvenience. Although neither cost, time, nor inconvenience may be small in obtaining a biological system model, yet, the information that can be obtained from the model generally requires less

time or cost, than information generated by constructing, testing, and measuring from the true conditions (Chestnut, 1965).

2. They may be used to investigate different scenarios before a suitable biological system is decided upon.
3. They can be manipulated to evaluate high-risk issues, such as economic repercussions, health effects, environmental damage, etc.
4. They may be used to generate alternative system improvements.
5. They provide an excellent means of communicating among the people (decision-makers, scientists, and system analysts) involved in the problem.
6. They offer a means of integrating the efforts of various disciplinary expertise, each of whom has different or partial interest in the problem.
7. They may be used to do research to identify responses or events which otherwise might be difficult due to the nature of the system.

#### **Modularity of mathematical models**

In order to develop the philosophical, theoretical, and conceptual foundations for the analysis of natural, biological, and human-made physical systems, it is recommended that:

1. The models of the processing components be modular—that is, models of subsystems such as plants, animals, soils, etc., are freestanding and self-contained, yet can be easily interconnected in a wide variety of ways to form a coherent model of the alternative biological system at all levels of vertical and horizontal integration.
2. The modules must be generic—that is, they must be transferable from one geographic location to another with appropriate changes in parameters and serve as building blocks for a wide variety of alternate biological systems.
3. The modular component models must deal with the economic, environmental, and technical aspects of a biological system in a coherent and systematic way.
4. The modular component models must be largely mathematical in nature so as to support both analytical investigations (as in policy and control analysis) as well as detailed computer simulations.
5. Insofar as possible, the modules should correspond to existing disciplines of applied biology, agriculture, forestry, natural resources, engineering, and other academic areas so that they can be developed,

maintained, and improved over time within the context of existing research and academic programs. In this respect the modeling efforts must be cumulative over time and academically dispersed among specialists who presumably know most about the substantive content of the modules.

### 3.5 Data Analysis

Models are formulated after a period of data collection on the true system. For example, Table 3.1 presents a data on the estimates of cumulative HIV infections worldwide from 1980 to 1996 (Brown et al., 1997, p. 85). The data is plotted and shown in Fig. 3.4. Table 3.2 and Fig. 3.5 present a data and graph, respectively, on the increasing electrical generating capacity of nuclear power plants (Brown, et. al., 1997, p. 49). In some cases, the data collected portrays the interaction of two or more biological systems. Table 3.3 and Fig. 3.6 present data and graph, respectively, on six-spotted mites and predatory mites (Ricklefs, 1993) interacting one another. Table 3.4 shows two populations, *Paramecium aurelia* and *Saccharomyces exiguus* (Gause, 1935). The graph in Fig. 3.7 shows that as the *P. aurelia* increases in population, the *S. exiguus* population drops. The drop in the population of the *S. exiguus* provides less food to the *P. aurelia* causing the latter species to decrease in population. The drop in the *P. aurelia* population allows the *S. exiguus* to recover and increase in population. The increase in the *S. exiguus* becomes favorable again to the *P. aurelia*.

A model of a biological system can be constructed in several methods. One method is the experimental or curve-fitting where a mathematical relationship or equation is identified by the closeness of fit with observed input-output data. Another method is the analytical technique, where basic biological and physical laws are systematically applied to the processing components and the interconnection of these components.

Table 3.1. Estimates of cumulative HIV infections worldwide (Brown et al, 1997, p.

Year	HIV infections
1980	0.2
1981	0.6
1982	1.1
1983	1.8
1984	2.7
1985	3.9
1986	5.3
1987	6.9
1988	8.7
1989	10.7
1990	13
1991	15.5
1992	18.5
1993	21.9
1994	25.9
1995	30.6

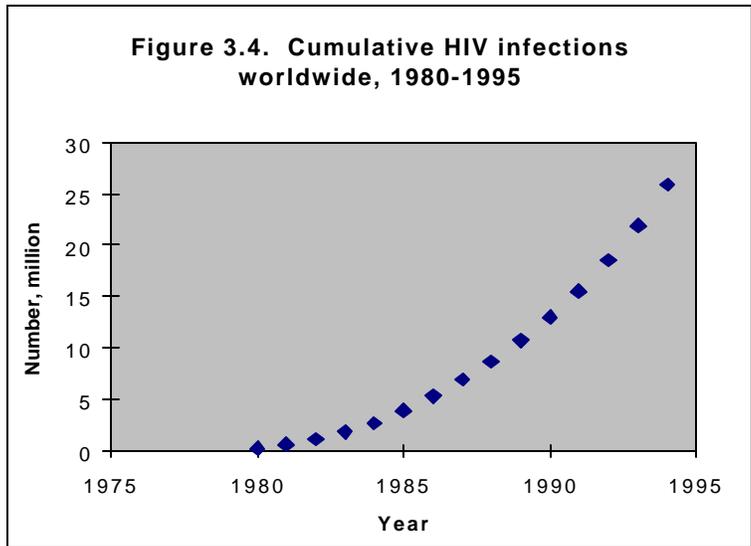


Table 3.2. Pesticide-resistant species of insects and mites Brown, et. al, 1997)

Year	No.
1908	1
1938	7
1948	13
1954	25
1957	75
1960	138
1963	156
1965	187
1967	225
1975	362
1978	400
1980	429
1984	449
1989	504

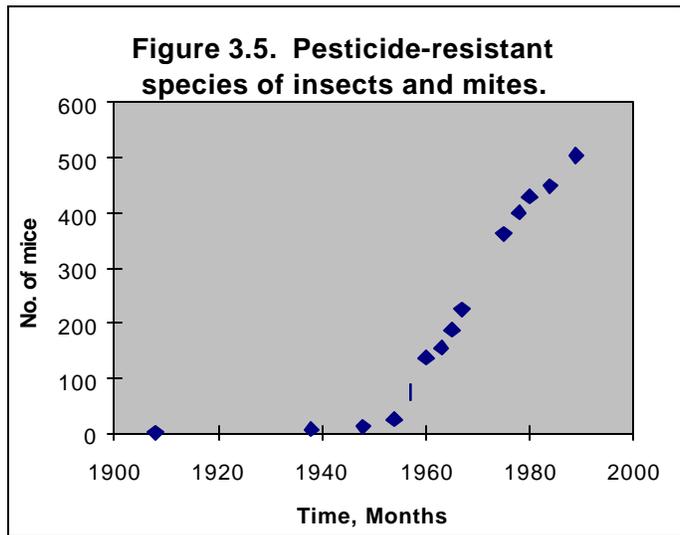


Table 3.3. Populations of six-spotted mites (SS) and predatory (P) mites (Ricklefs, 1993).

Days	SS mites	P mites
0	100	100
10	455	175
20	1300	1300
25	1100	1900
30	545	1560
40	225	688
50	250	360
60	306	260
70	445	107
80	1123	156
90	1663	576
100	2069	998
110	1545	1130
120	713	1996
130	814	218
140	898	516
150	1004	123
160	2078	148
170	1874	678
180	1666	1007
190	1163	2360
200	246	865

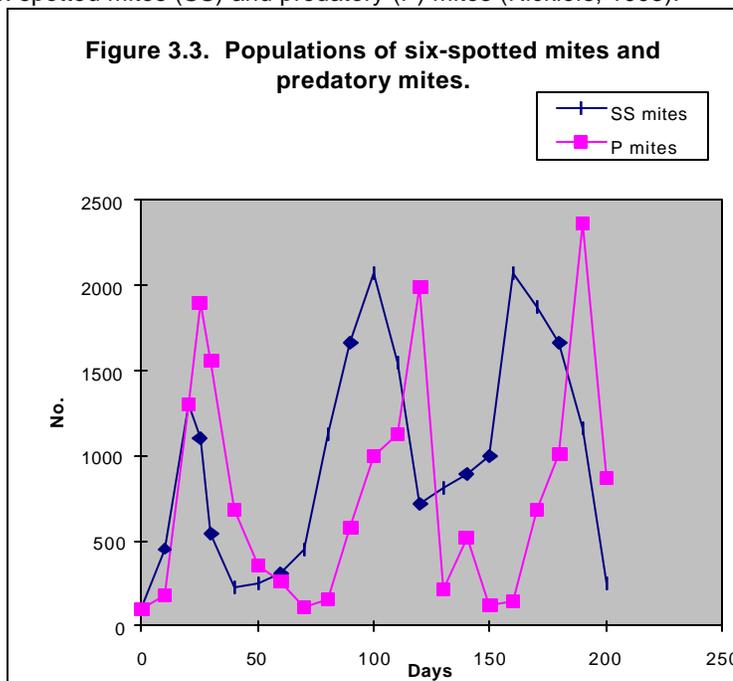
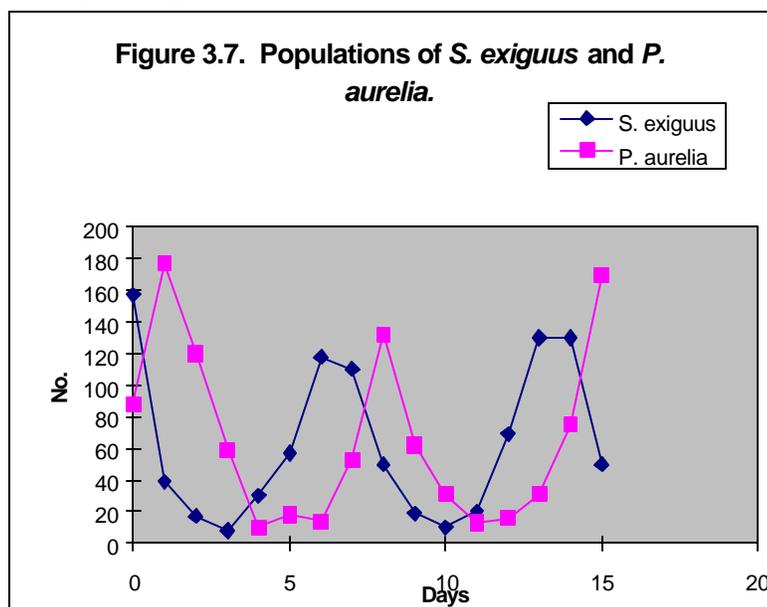


Table 3.4. Populations of *Saccharomyces exiguus* and *Paramecium aurelia* (Gause, 1935).

Days	<i>S. exiguus</i>	<i>P. aurelia</i>
0	157	88
1	39	177
2	17	120
3	8	59
4	30	10
5	57	18
6	118	14
7	110	53
8	50	132
9	19	62
10	10	31
11	20	13
12	69	16
13	130	31
14	130	75
15	50	170



### 3.6 Steps in Biological systems Modeling

The steps to develop a mathematical model are iterative in nature, that is, the modeling process may return several times to the various steps before the model is in final form. The steps in biological system modeling are outlined as follows:

1. State the specific problem to be modeled.
2. Identify the specific purpose of the model.
3. Define the system components.
4. Define the inputs, outputs, technological parameters, boundary conditions, and evaluation criteria for each of the component and for the whole system, whenever necessary.
5. Propose a topological structure through the component interconnections. Describe the spatial and time domain structures that best represent the system.
6. Describe mathematically each material transformation, transportation, and storage processes. Identify energy functions associated with each material flow rate and transformation processes, if necessary.
7. Develop the biological system model equations.
8. Test if the biological system model is useful and relevant.
9. If the biological system model is relevant and useful, validate the model. If the biological system model is not useful, repeat the modeling process.
10. Modify the biological system model if necessary.

Steps 1-10 may be repeated several times until a satisfactory model is developed. Fig. 3.8 illustrates the steps in graphical form. One critical factor in developing the model is the identification of the system boundary. The boundary separates the system under study and the rest of the system (the environment). The separation could be physical or abstract. A requirement for the selection of the system boundary is that all signals crossing the boundary should be one-way directions, that is, the system output should not affect the environment to the extent that it would modify the signals from the environment to the system (Eisen, 1988).

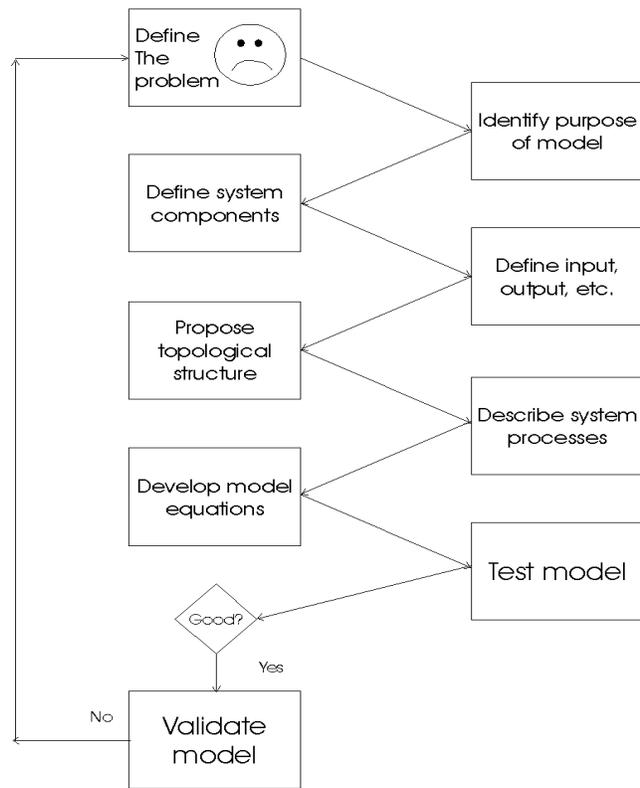


Figure 3.8. Graphical illustration of the steps in modeling.

### Example 31. Developing a World Population Model

The steps in modeling will be illustrated in the following hypothetical situation. The United Nations has requested the Food and Agriculture Organization to present a plan for meeting the world food requirement in the year 2000.

#### Problem statement

In order to provide enough food for all, food demand needs to be estimated. The first step toward this end is to estimate the world population in the year 2000.

#### Purpose of the model

The purpose of the model is to estimate the world population in the year 2000, given a world population data for 1986.

#### System components

Components: Countries of the world.

Sub-components:

1. Population size of each country, aggregated to represent the world population size.
2. Population growth rate of each country, averaged to represent the world population growth rate.

### **System boundary**

Planet earth is the system; everything outside of it is the environment.

### **Input/output**

No immigration or emigration of people from and to other planets outside of the earth.

### **Proposed topological structure**

Simple aggregation of all population sizes and growth rates to represent the world population size and average growth rate, as shown in Fig. 3.9. The following notations are introduced to develop the world population model:

$t$  = a continuous variable denoting time

$\Delta t$  = interval of time between  $t_1$  and  $t_2$

$N(t)$  = world population size at time  $t$

$b$  = average birth rate (fraction of world population)

$d$  = average death rate (fraction of world population)

### **Developing the population model**

Assumption: On the average,  $b$  and  $d$  are constant over the interval of time  $\Delta t$ .

1. The increase in world population size in the interval of time  $\Delta t$  is:  $b N(t) \Delta t$ .
2. The decrease in world population size in the interval of time  $\Delta t$  is:  $-d N(t) \Delta t$
3. The net change in population size in the interval of time  $\Delta t$  is given as:

$$N(t + \Delta t) - N(t) = \Delta N = (b N(t) - d N(t)) \Delta t$$

4. Simplifying the above equation, we have:

$$\Delta N = r N \Delta t \tag{3.4}$$

where  $r$  is the net growth rate and  $N$  is a function of time  $t$ .

5. Dividing both sides of Equation (3.4) by  $\Delta t$ , we have:  $\frac{\Delta N}{\Delta t} = rN$

6. Taking the limit as  $\Delta t \rightarrow 0$ , we get the derivative of N with respect to t:

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta N}{\Delta t} = \frac{dN}{dt} = \dot{N}(t)$$

And so, Equation (3.4) becomes:  $\frac{dN}{dt} = rN$   
(3.5)

Equation (3.5) can be solved by separating the variables, that is, by grouping all variables related to N on the left-hand side of the equality sign and all variables related to time t on the right-hand side of the equality sign.

7. Separating the variables in Equation (3.5) results in:

$$\frac{dN}{N} = r dt \tag{3.6}$$

8. Integrating Equation (3.6) from  $t=0$  (1986) to  $t=t_1$  (2000), we have:

$$\int_0^{t_1} \frac{dN}{N} = r \int_0^{t_1} dt \tag{3.7}$$

9. If we generalize  $t_1$  to be any time t, the solution to Equation (3.7) is:

$$\ln \frac{N(t)}{N(0)} = rt \tag{3.8}$$

where  $\ln$  denotes the natural logarithm function.

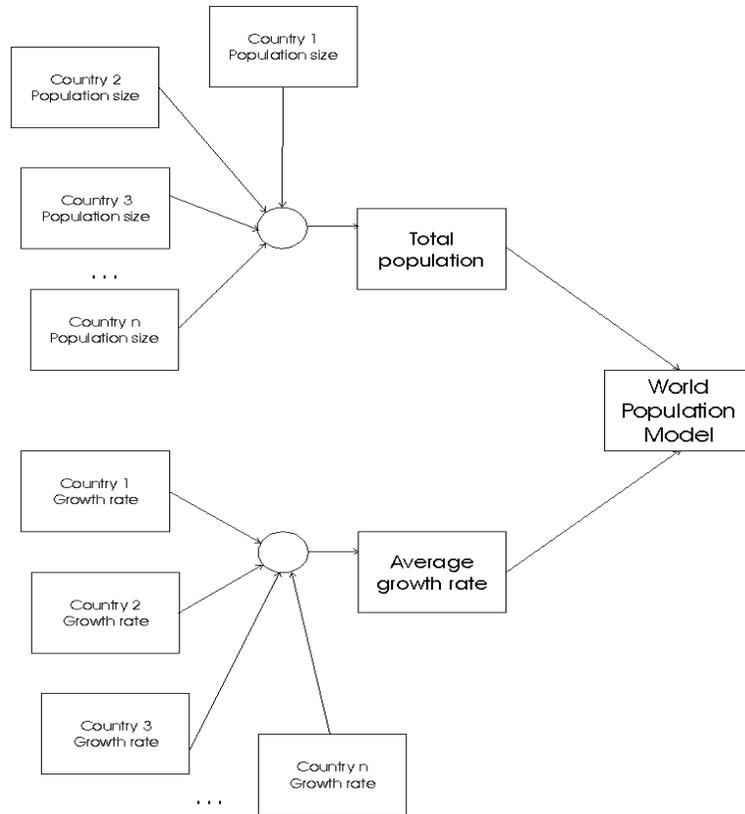


Figure 3.9. Topological structure of a world population model.

10. Exponentiating both sides of Equation (3.8) and using the relationship  $e^{\ln x} = x$ , Equation (3.8) becomes the exponential growth equation

$$N(t) = N(0)e^{rt} \quad (3.9)$$

where  $N(0)$  is the initial population at time zero. If the birth rate is less than the death rate,  $r$  becomes negative. Instead of a population growing, the population is decaying or decreasing over time in proportion to its size. In this case, we can rewrite (3.9) as:

$$\frac{dN}{dt} = -rN \quad (3.10)$$

The solution to Equation (3.10) is the exponential decay function

$$N(t) = N(0)e^{-rt} \quad (3.11)$$

Growth and decay processes are characteristics of biological systems, such as population growth, increasing pollution, growth of cancer cells, increasing political unrest, increasing civil disorder, increasing poverty, decreasing natural

resources, and decreasing water quality. Whenever there is a change, it can always be regarded as a growth or decay process.

The net growth rate  $r$  can be estimated from historical data, such as from the world population data of 1900-1995 given in Table 3.5 and the curve shown in Fig. 3.10 (Harte, 1985, p.263; Brown et al., 1997, p. 81). If we take the natural logarithm of both sides of Equation (3.9), we get

$$\ln N(t) = \ln N(0) + rt$$

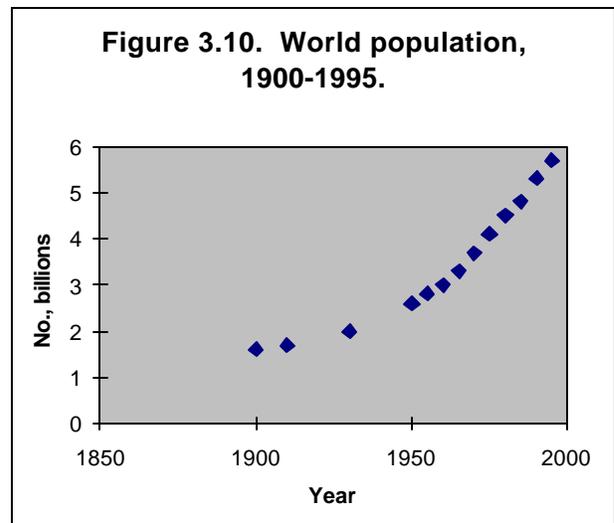
which is a straight line equation with  $\ln N(0)$  as the intercept and  $r$  as the slope. The growth rate  $r$  can now be calculated from the usual formula for the slope of a straight line given as follows:

$$r = \frac{\ln N(t_2) - \ln N(t_1)}{t_2 - t_1}$$

where  $N(t_1)$  and  $N(t_2)$  are the ordinates on the straight line determined from the population data.

Table 3.5. World population, 1900-1995.

Year	No. in billions
1900	1.6
1910	1.7
1930	2
1950	2.6
1955	2.8
1960	3
1965	3.3
1970	3.7
1975	4.1
1980	4.5
1985	4.8
1990	5.3
1995	5.7



### Validation of the world population model

According to Archer et al. (1987, p. 45), the world population in 1980 was about 4.432 billion growing at a rate of 1.7%. If we take 1980 as  $t=0$ , Equation (3.9) can now be written as  $N(t) = 4.432e^{0.017t}$ . Then the estimated world population in 1986 ( $t=6$  years) is  $N(6) = 4.432e^{0.017*6}$  or 4.908 billion people. According to Bittinger and Morrel (1988, p. 269), the world population in 1986 was 4.9 billion. We can either accept or reject our population model. For our own purpose, we shall accept the model as relevant, useful, and validated.

### Using the model to estimate the world population

We can now use Equation (3.9) to estimate world population in the year 2000 using Bittinger and Morrel's world population data for 1986, which is 4.9 billion. During that time, the net growth rate was estimated to be 1.6% per year (Bittinger and Morrel, 1988, p. 269). Equation (3.9) can now be written as  $N(t) = 4.9e^{0.016t}$ . Therefore, the estimated world population in the year 2000 is 6.13 billion. According to Archer et al. (1987, p. 45), the estimated world population by the year 2000 is 6.10 billion.

### Example 3.2. Developing a Lake Pollution Model, Part I

Consider a problem where 50 kg of a pollutant P was dumped into a lake once at initial time. Assume that the pollutant is stable, highly soluble, and uniformly mixed in the lake. The lake volume is given to be  $4 \times 10^7 \text{ m}^3$  and the average water flow-through rate is  $5 \times 10^4 \text{ m}^3 \text{ d}^{-1}$ . What is the pollutant concentration after 10 days?

The following notations are introduced:  $t$  is a continuous variable denoting time;  $\Delta t$  is the small interval of time;  $C$  is the pollutant concentration,  $\text{kg m}^{-3}$ ;  $P$  is the amount of pollutant, in kg;  $F_W$  is the flow rate of water, in  $\text{m}^3 \text{ d}^{-1}$ ; and  $V$  is the volume of water, in  $\text{m}^3$ .

Problem statement: To find the pollutant concentration over time,  $C(t)$ .

Objective of the model: To be able to estimate the pollutant concentration at any given time of a lake with the following characteristics: 50 kg of the pollutant is dumped once at initial time,  $t=0$ ; the water flow rate  $F_W$  is  $5 \times 10^4 \text{ m}^3 \text{ d}^{-1}$ , and the lake volume  $V$  is  $4 \times 10^7 \text{ m}^3$ .

System components: The system includes the lake, inlet stream, outlet stream, and pollutant.

Topological structure: The topological structure is illustrated in Fig. 3.11.

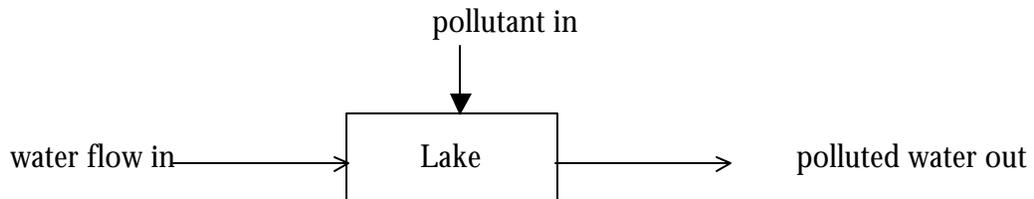


Figure 3.11. A topological structure of lake pollution.

System processes:

- Pollutant is stable (does not disintegrate into other forms) and does not volatilize.
- Pollutant is highly soluble (immediately mixed in water upon contact).
- Pollutant is uniformly mixed in water.
- Water flow in the inlet stream is the same rate as water flow in the outlet stream.

Develop model equations:

1. The change in the amount of pollutant ( $\mathbf{DP}$ ) in the interval of time  $\mathbf{Dt}$  is:

$$\mathbf{DP} = (\text{inflow rate} - \text{outflow rate}) \mathbf{Dt}$$

inflow rate = 0, since the pollutant is dumped only once at time 0.

outflow rate =  $-F_w C$ , the negative sign means a decrease in the amount.

Therefore, the rate of change of the pollutant in the interval of time  $\mathbf{Dt}$  is

$$\mathbf{DP} = (0 - F_w C) \mathbf{Dt}$$

2. The change in pollutant concentration ( $\mathbf{DC}$ ) in the interval of time  $\mathbf{Dt}$  is the change in the amount of pollutant divided by the lake volume  $V$ :

$$\mathbf{DC} = \frac{\mathbf{DP}}{V} = \frac{-F_w C}{V} \mathbf{Dt}$$

3. Dividing both sides of the equation by  $\mathbf{Dt}$ , we have  $\frac{\mathbf{DC}}{\mathbf{Dt}} = \frac{-F_w}{V} C$

4. Take the limit as  $\mathbf{Dt} \rightarrow 0$ , we have:  $\lim_{\mathbf{Dt} \rightarrow 0} \frac{\mathbf{DC}}{\mathbf{Dt}} = \frac{dC}{dt}$

The differential equation is in the form:  $\frac{dC}{dt} = -\frac{F_w}{V} C$

5. Solving the differential equation by separating the variables and taking the integral:  $\int_0^{t_1} \frac{dC}{C} = -\frac{F_w}{V} \int_0^{t_1} dt$  results in the equation  $\ln \frac{C(t)}{C(0)} = -\frac{F_w}{V} t$ , where  $t=t_1$ .

6. Exponentiating both sides of the equation results in the lake pollution model having the form:

$$C(t) = C(0)e^{-\frac{F_w t}{V}}$$

7. The pollutant concentration at  $t=0$  is:  $C(0) = \frac{P}{V}$

8. The final form of the lake pollution model is:  $C(t) = \frac{P}{V}e^{-\frac{F_w t}{V}}$

9. The pollutant concentration after 10 days is:

$$C(t) = \frac{50kg}{4 \times 10^7 m^3} e^{-\frac{5 \times 10^4 m^3 d^{-1}}{4 \times 10^7 m^3} 10d} = 1.27 \times 10^{-6} kg m^{-3}$$

### Example 3.3. Developing a Discrete Population Model

A population can also be estimated using a discrete system of equations. Take a population of opossums, for example. An opossum is a small, omnivorous, tree-dwelling animal of the marsupial family. Suppose, an opossum population has an initial size of 100 and increases by 25% per year. What is the opossum population after 10 years?

Let  $y(k)$  be the size of the population at the beginning of year  $k$ . Intuitively, the population model can be written as follows:

$$\text{population in year } k = \text{population in year } (k-1) + \text{population increase during year } (k-1)$$

The model can be written as follows:

$$y(k) = y(k-1) + 0.25 y(k-1)$$

which leads to the equation:  $y(k) = 1.25 y(k-1)$ .

We are given the initial population  $y(0) = 100$ . To find the population after 10 years, begin by solving iteratively from year 1 up to year 10 by proceeding as follows:

$$y(1) = 1.25 y(0)$$

$$y(2) = 1.25 y(1) = 1.25 [1.25 y(0)] = 1.25^2 y(0)$$

$$y(3) = 1.25 y(2) = 1.25^3 y(0)$$

From this iterative procedure, we can develop a general formula for the opossum population as follows:

$$y(k) = 1.25^k y(0).$$

And so, after 10 years, the opossum population is:  $y(10) = 931$ .

**Example 3.4 Discrete Population with a Step Forcing Function**

If 10 of the opossums are poached each year, find the size of the population after 10 years. Intuitively, the population model can be written as follows according to the assumption:

$$\frac{\text{change in population}}{\text{size in year } k} = \frac{\text{population size}}{\text{increase in year } k} - \frac{\text{number of}}{\text{poached opossums}}$$

Quantitatively, the above expression can be translated into:

$$y(k) = y(k-1) + 0.25 y(k-1) - 10$$

which reduces to:  $y(k) = 1.25 y(k-1) - 10$ . We proceed again by solving iteratively as follows:

$$y(1) = 1.25 y(0) - 10$$

$$y(2) = 1.25 y(1) - 10 = 1.25 [1.25 y(0) - 10] - 10 = 1.25^2 y(0) - 1.25 * 10 - 10$$

$$y(3) = 1.25 y(2) - 10 = 1.25 [1.25^2 y(0) - 1.25 * 10 - 10] = 1.25^3 y(0) - 1.25^2 * 10 - 1.25 * 10 - 10$$

The general formula at any time k is as follows:

$$y(k) = (1 + r)^k y(0) - 10 \sum_{i=0}^{k-1} (1 + r)^i$$

**Example 3.5. Discrete Population with a Ramp Forcing Function**

Suppose the number of poached opossum increases by 5 per year, what is the population size after 10 years? The population model will now have the following form

$$y(k) = y(k-1) + 0.25 y(k-1) - 10 - 5(k-1)$$

where the component  $(-10 - 5(k-1))$  is called a ramp-type forcing function. The iterative procedure will look like this:

$$y(1) = 1.25 y(0) - 10 - 5 * 0$$

$$y(2) = 1.25 y(1) - 10 - 5 \cdot 1 = 1.25 [1.25 y(0) - 10 - 0] - 10 - 5 = 1.25^2 y(0) - 1.25 \cdot 10 - 10 - 5$$

$$y(3) = 1.25 y(2) - 10 - 5 \cdot 2 = 1.25 [1.25^2 y(0) - 1.25 \cdot 10 - 10 - 5] - 10 - 5 = 1.25^3 y(0) - 1.25^2 \cdot 10 - 1.25 \cdot 10 - 1.25 \cdot 5$$

After 10 years, the opossum population is estimated to be 134. The general formula at any time  $t$  is given as follows:

### 3.7 System Classification

Systems are classified according to the mathematical equations used to describe them. Systems can be classified either as lumped-parameter or distributed-parameter, deterministic or stochastic, continuous or discrete, linear or nonlinear, stationary or nonstationary, homogeneous or nonhomogeneous (Eisen, 1988, pp. 29-49). In this course, we shall be studying biological system models which can be characterized as lumped-parameter, deterministic, continuous, linear and nonlinear, stationary, homogeneous and nonhomogeneous systems.

#### **Lumped-Parameter vs. Distributed-Parameter Systems**

Each element in a lumped-parameter system can be treated as if it were concentrated (“lumped”) at one particular point in the system. The position of the point can change with time. A system of this type can be analyzed using ordinary differential or difference equations. On the other hand, elements in a distributed-parameter system are distributed in space (distance, area, or volume). Their variables must be treated as changing not only from time to time but also from point to point. This type of system can be analyzed using partial differential or difference equations since the variables vary with time and space.

A classic example of a lumped-parameter system is human population growth, as described in Equation (3.9). We can assume population growth as a process where elements of the system are concentrated (“lumped”) at a particular location (city, country, or world) and varying over time. Another example of a lumped-parameter system is the exponential decay function in Equation (3.10).

#### **Example 3.6 Human population growth**

The population of the United States was 241 million in 1986 (Bittinger and Morrel, 1988, p. 270) and estimated to be growing exponentially at a rate of 0.9% per year. The population model is written as  $N(t) = 241e^{0.009t}$ . The estimated U.S. population in 1994 ( $t=8$ ) is 259 million. What is the estimated U.S. population in the year 2000 if the trend continues?

#### **Deterministic vs. Stochastic Systems**

A system is deterministic when the response to a specified input applied under specific conditions is completely predictable from physical laws. The

response behavior of this type of system is reproducible at any time. In stochastic systems, it is not possible to predict which output will occur at any time although a set of all possible outputs for each given input may be known. In this case, probability theory is required to predict the output.

The U.S. population model in Example 3.6 is an example of a deterministic system—the output can be predicted precisely at any point in time. Suppose an unknown virus suddenly caused an epidemic leading to an unpredictable growth behavior over time. In this case, the growth rate  $k$  becomes a random variable (stochastic in nature) and therefore, the size of the U.S. population at any time  $t$  cannot be predicted.

### **Continuous vs. Discrete Systems**

In a continuous system, all the important variables are continuous in nature and the time that the system operates is some interval of the real numbers. This type of system can be described by differential equations. Example 3.6 is an example of a continuous system. In a discrete system, all important variables are discrete and the time that the system operates is a finite or countably infinite subset of the real numbers. System of this type can be described by difference equations.

### **Example 3.7. Discrete model of the U.S. population**

In Example 3.6, the U.S. population in 1994 was estimated using equation (3.9), a continuous exponential growth model. The U.S. population can also be estimated by using the discrete model equivalent of equation (3.9) given in the form:

$$N(k \mathbf{D}) = [1+r]^k N(0) \quad (3.12)$$

where  $k$  is the sampling frequency,  $\mathbf{D}$  is the sampling interval,  $r$  is the specific growth rate, and  $N(0)$  is the initial population. If the sampling interval  $\mathbf{D}$  is 1 year, then the discrete population model of the U.S. population can now be given as follows:  $N(k) = [1 + 0.009]^k 241$ . To estimate the population in 1994, the sampling frequency  $k$  is 8 and the result will be:  $N(8) = 259$  million. It gives a similar result as the continuous model in Example 3.6.

### **Linear vs. Nonlinear Systems**

A system is linear if a change in input level causes a proportionate change in the output. Linearity is formally defined as follows: suppose that the admissible inputs  $u_1$  and  $u_2$  to a system produce  $y_1$  and  $y_2$ . Then the system is linear if, and only if, the input  $a_1 u_1 + a_2 u_2 = a_1 y_1 + a_2 y_2$  for any real numbers  $a_1$  and  $a_2$ .

There are many general theorems, techniques, and approaches to study the characteristics and properties of linear systems. Many nonlinear systems can be approximated by linear systems for certain values of their variables and the resulting linear systems can be analyzed by powerful linear systems techniques.

A system that is not linear is nonlinear. The exponential population model shown in Example 3.6 is an example of a linear system. A more realistic population model can be formulated where growth is not indefinite. There are many reasons why population can never grow beyond a limiting value. The inhibited growth could be due to lack of space, decreased available food, reduced oxygen supply, presence of toxic substances, and other limiting factors. In such situations, growth rate decreases as population size increases, and approaches zero as population size approaches the maximum limiting value. The inhibited growth model is expressed as follows:

$$\frac{dN}{dt} = N'(t) = rN(L - N) \quad (3.13)$$

where  $L$  is the limiting value and  $r > 0$ . Equation (3.13) is called the logistic function. The solution to equation (3.13) is:

$$N(t) = \frac{N_0 L}{N_0 + e^{-Lrt} (L - N_0)} \quad (3.14)$$

Equation (3.14) has an inflection point at that value of  $t$  where  $N(t) = 1/2 L$ . The inflection point is that point when a curve changes concavity (Bittinger and Morrel, 1988, p. 555). It gives the time and population size at which the rate of population growth reaches a maximum and changes from an increasing function to a decreasing function.

### **Example 3.8 Estimating the pheasant population**

A pheasant population data (Bittinger and Morrel, 1988, pp. 555) and its graphical form are shown in Table 3.6 and Fig. 3.9, respectively.

From the data, the pheasant population can be modeled using the logistic function of equation (3.14) with the following parameters:  $N_0$  (initial population) = 6;  $L$  (maximum population) = 2886;  $r$  (specific growth rate) = 0.0003357; The logistic population model can now be written as follows:

$$N(t) = \frac{6 * 2886}{6 + e^{-2886 * 0.0003357 * t} (2886 - 6)}$$

A graph of the data points from 1937 through 1943 and a graph of the population estimate from 1937 through 1950 are presented in Fig. 3.10.

Table 3.6. Pheasant population

Time	Population
1937	8
1938	30
1939	81
1940	282
1941	641
1942	1194
1943	1898

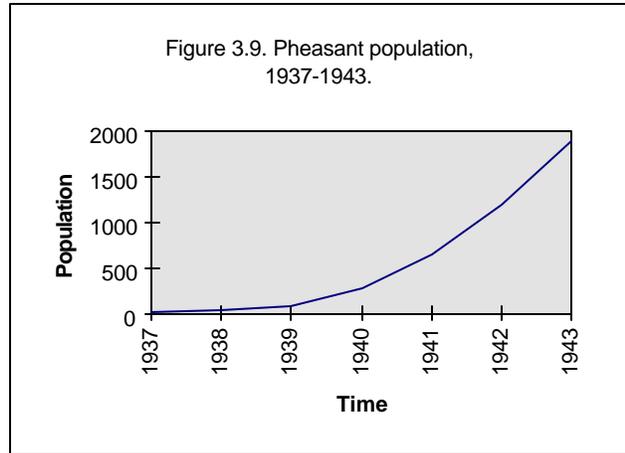
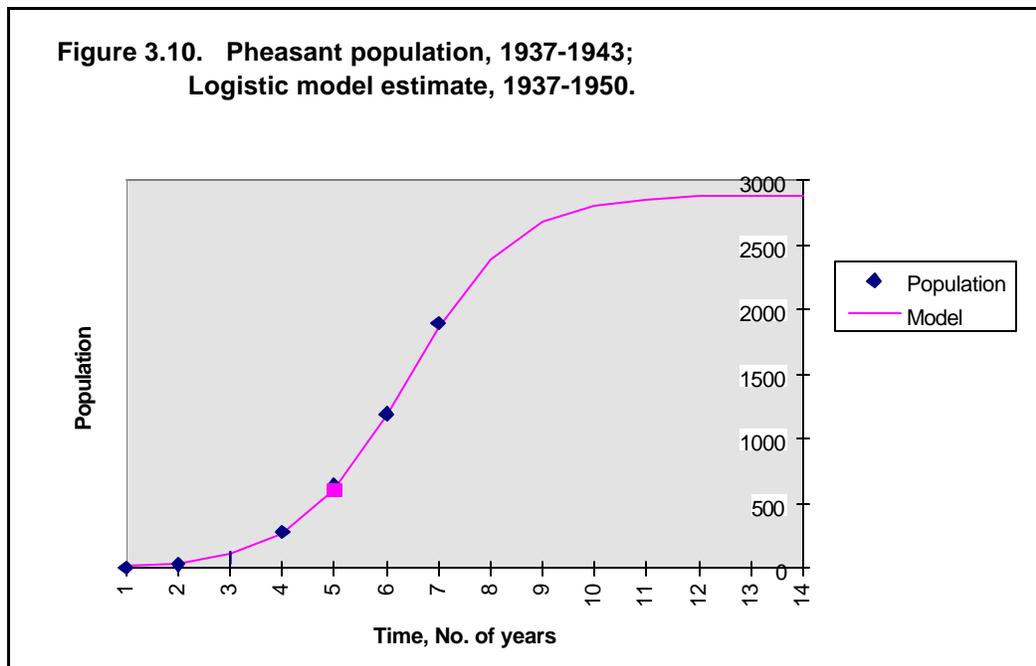


Figure 3.10. Pheasant population, 1937-1943; Logistic model estimate, 1937-1950.



### Stationary vs. Nonstationary Systems

A stationary system is also known as time-invariant. The form of the equations characterizing the system do not vary with time when the inputs are constant. A nonstationary system is also known as time-varying. The form of the equations characterizing the system vary with time. All our examples so far are stationary—the specific growth rate  $r$  is not a function of time.

### Homogeneous vs. Nonhomogeneous Systems

A homogeneous system is also known as an unforced system—there are no external inputs and the system behavior is determined entirely by its initial

conditions. A nonhomogeneous system is also known as a forced system—there are external inputs to the system. A nonhomogeneous system of the logistic growth model is given as follows:

$$dN/dt = N'(t) = rN(L-N) + u(t) \quad (3.15)$$

where  $u(t)$  is the input or forcing function. Table 3.7 shows some models of populations and their respective classification.

### 3.8 Input Functions of Time

One major characteristic of biological systems is that they behave in a manner that is a function of time. Time is the independent variable to which all system variables can be correlated. In order to analyze a biological system, one must understand the different kinds of time functions which are characteristic of system variables and system inputs. Input time functions can be classified as either deterministic or random. In this course, we shall be dealing with deterministic input time functions.

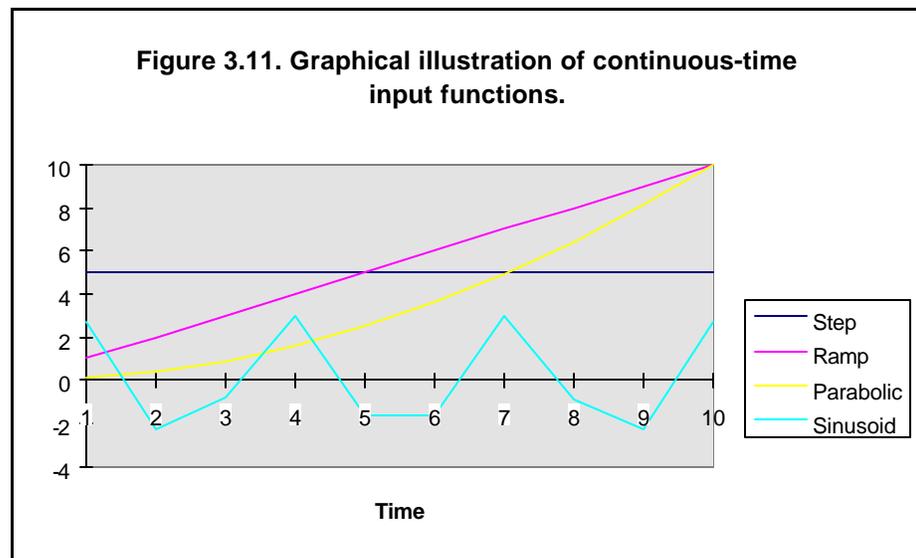
Earlier, we discussed about nonhomogeneous systems which have forcing functions or input functions, such as the logistic growth model with input, expressed through equation (3.15). Every input which is characterized by a deterministic time function has a unique output which is determined by the system properties. The class of deterministic input time functions presented below are the step, ramp, parabolic, and sinusoidal. The mathematical forms of the input functions are presented in Table 3.8. Figure 3.11 illustrates the four deterministic, continuous input functions.

Table 3.7. Some examples of biologically-based systems and their classification.

Common Name	Mathematical Representation	Classification
Exponential growth function	$N'(t) = rN$	lumped-parameter, deterministic, continuous time, linear, stationary, homogeneous
Exponential decay function	$N'(t) = -rN$	Same classification as the exponential growth function.
Exponential growth discrete function	$N(k) - N(k-1) = rN(k-1)$ or $N(k) = [1 + r] N(k-1)$	lumped-parameter, deterministic, discrete time, linear, stationary, homogeneous
Logistic or Verhulst function	$N'(t) = rN(L - N)$ or $N'(t) = rN(1 - N/L)$	lumped-parameter, deterministic, continuous time, nonlinear, stationary, homogeneous
Logistic function w/ input	$N'(t) = rN(1 - N/L) + u$	lumped-parameter, deterministic, continuous time, nonlinear, stationary, nonhomogeneous
Exponential growth discrete function w/ input	$N(k) - N(k-1) = rN(k-1) + u(k)$	lumped-parameter, deterministic, discrete time, linear, stationary, nonhomogeneous
Lotka-Volterra model or host-parasite system	$H'(t) = aH - cHP$ $P'(t) = -bP + dHP$	lumped-parameter, deterministic, continuous time, nonlinear, stationary, homogeneous
Predator-prey system	$N_1'(t) = rN_1(1 - N_1/L) - aN_1N_2$ $N_2'(t) = -cN_2 + bN_1N_2$	lumped-parameter, deterministic, continuous time, nonlinear, stationary, homogeneous
Competition system	$N_1'(t) = r_1N_1(1 - N_1/L_1) - aN_1N_2$ $N_2'(t) = r_2N_2(1 - N_2/L_2) - bN_1N_2$	(similar classification as the predator-prey system)

Table 3.8. Mathematical form of the input functions of time.

Input function	Mathematical form
Step	$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$
Ramp	$u(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$
Parabolic	$u(t) = \begin{cases} 0, & t < 0 \\ t^2, & t > 0 \end{cases}$
Sinusoidal	$u(t) = \begin{cases} 0, & t < 0 \\ \mathbf{a} \sin(\mathbf{v}t + \mathbf{f}), & t \geq 0 \\ \mathbf{a} \cos(\mathbf{v}t + \mathbf{f}), & t \geq 0 \end{cases}$ <p>where <math> \mathbf{a} </math> is the amplitude; <math>2\mathbf{p} / \mathbf{v}</math> is the period; and <math>\mathbf{f} / \mathbf{v}</math> is the phase shift.</p>



### Example 3.9. The U.S. population with immigration

In Example 3.6, we modeled the U.S. population with an exponential growth function in order to predict its population in 1994 from a base population size of 241 million in 1986 and a specific growth rate of 0.9% per year. We assumed that there were no input to the system, that is, no immigration (positive input) nor emigration (negative input) was allowed. However, that assumption may not be realistic because the U.S. foreign policy does allow for immigration and emigration. Suppose, the U.S. would allow 500,000 people to come in and live here beginning 1987. This is a situation where the input is a step function and  $u(t) = 0.5$  million. The exponential growth equation can be formulated in the following manner:

$$N'(t) = 0.009N + 0.5, \quad N(0) = 241 \text{ million}$$

The general solution to the above differential equation is:

$$N(t) = ce^{0.009t} - \frac{0.5}{0.009}$$

From the initial condition, we can solve for  $c$  and the general solution is given as

$$N(t) = 297e^{0.009t} - 56$$

From this equation, the predicted U.S. population in the year 2000 (assuming that the trend will continue) under the assumption of a step-function immigration input is about 281 million (compared to 273 million without immigration). Using Mathematica, show a comparison of the predicted U.S. population from 1987 to the year 2000 without immigration and with a step input function.

Suppose that the U.S. Immigration and Naturalization Service (INS) has adopted the policy that after 1986, the number of people to be allowed in the U.S. will be 510,000 for the year 1987 and increasing by 10,000 every year thereafter. That is, 510,000 will be allowed in 1987; 520,000 in 1988; 530,000 in 1989, and so on. This is now a case where the input is a ramp function and  $u(t) = 0.5 + 0.01t$  million. The exponential population model can be written as follows:

$$dN/dt = 0.009 N(t) + 0.5 + 0.01t, \quad N(0) = 241 \text{ million}$$

Using this equation and assuming that the trend will continue, the predicted U.S. population in the year 2000 is about 283 million.

## 3.9 Output Functions of Time

In general, the output functions of time are not the same as the input functions of time. The output functions depend on the properties of the system. The output functions of Equation 3.9 is also known as the exponential growth model. The output function of Equation 3.13 is also known as the logistic

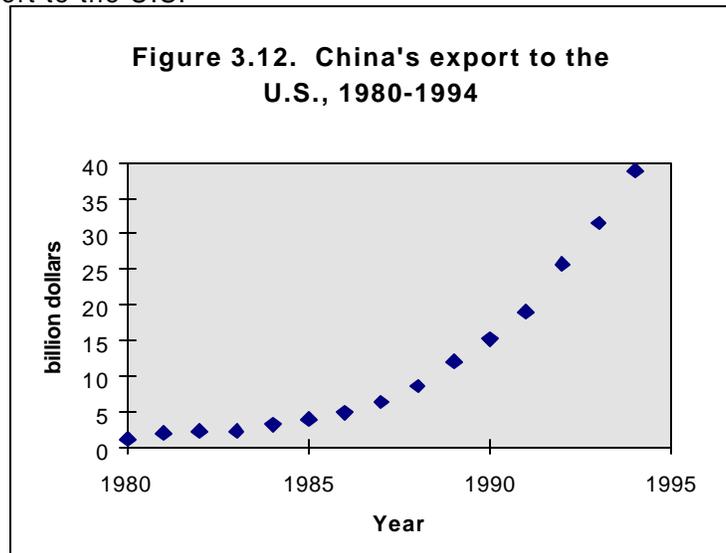
function. In general, the response curve of biological systems is characterized by the function  $e^x$ , where  $x$  can be any variable or group of variables and can be positive or negative valued.

## Exercises

1. *Business: China's export to the U.S.* Develop a mathematical model for China's export to the US. Validate your model based on the data presented Table 3.9 and Fig. 3.13 (Brown, 1995, p. 104). Based on your model and assuming that the trend will continue, how much is China's expected export to the U.S. in 2010?

Table 3.9. China's export to the U.S.

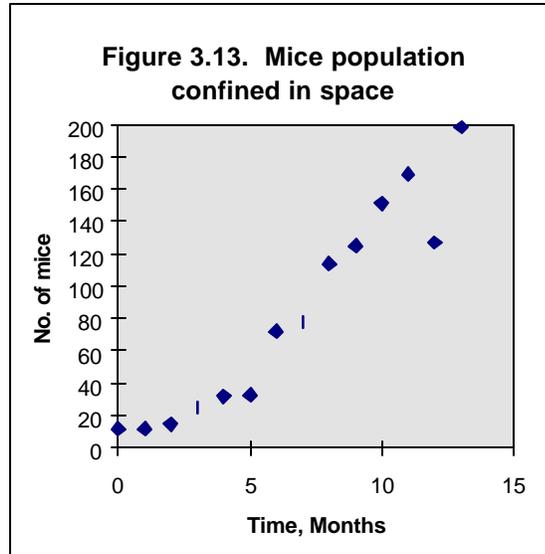
Year	Export
1980	1.1
1981	1.9
1982	2.2
1983	2.2
1984	3.1
1985	3.9
1986	4.8
1987	6.3
1988	8.5
1989	12
1990	15.2
1991	19
1992	25.7
1993	31.5
1994	38.9



2. *Ecology: Mice growth.* Using a spreadsheet program, find a best-fit curve model to describe the growth of mice in a confined area, as given in Table 3.10 and shown in Fig. 3.14 (Snyder, 1976; Murray, 1979).

Table 3.10. Mice population

Months	No. of Mice
0	12
1	12
2	15
3	25
4	32
5	33
6	72
7	78
8	114
9	125
10	151
11	169
12	127
13	198



3. *Ecology: Population growth, I.* Show the population growth curve over a period of 20 years using the exponential growth equation when the growth rate is 0.9% and the initial population is 241 million.
4. *Ecology: Population growth, II.* Show the population growth curve of Exercise no. 3 when 56 people leave the system yearly.
5. *Ecology: Population growth, III.* Show the population growth curve of Exercise no. 3 using the discrete population model. Compare output with graph in Exercise no. 3. Discuss your observation.
6. *Ecology: Population of spider mite Willamette.* The daily population size of spider mite Willamette is given as follows: day 0, 8 mites; day 10, 7 mites; day 20, 10 mites; day 30, 12 mites; day 40, 12 mites; day 50, 20 mites; day 60, 50 mites; day 70, 150 mites; day 80, 230 mites; and day 90, 250 mites (Flaberty and Huffaker (1970). Plot the original data. Find a model that may be used to describe the population growth. Plot the model. Put the two plots together. Does your model describe well the population growth?
7. *Natural Resources: Petroleum Consumption.* Petroleum consumption follows an exponential function as in equation (3.9). In 1980, petroleum consumption was  $1.35 \times 10^{20}$  J/yr (Harte, 1985, p. 112). The petroleum consumption increases by 2% every year. What is the petroleum consumption in 1981? What is the petroleum consumption in the year 2000?
8. *Business: Cost of a TV Commercial.* According to Bittinger and Morrel (1988, p. 272), a TV commercial costs \$80,000 in 1967, \$200,000 in 1970, \$324,000 in 1977, \$550,000 in 1981, \$800,000 in 1983, and \$1,100,000 in 1985. The cost of a 60-sec TV commercial follows the exponential function of equation (3.9). If we let  $N(t)$  to be the cost at any time  $t$ ,  $t=0$  to be 1967, and  $N(0) = \$80,000$

be the cost at time  $t=0$  (1967), then (a) write the equation with the initial conditions, (b) find the growth rate  $r$  using the data points between 1967 and 1977 ( $t=10$ ), and (3) using the calculated  $r$  value, estimate the cost of a 60-sec commercial in 1994.

9. *Ecology: Population of Developed Countries.* According to Archer et al. (1987, p. 45), the population of the developed countries in 1980 was about 1130 million with an annual growth rate of 0.7%. What is the population of the developed countries by the year 2000? After what period of time will the population of 1980 be doubled?
10. *Biomedical: Alcohol Absorption and Associated Risk.* According to Bittinger and Morrel (1988, p. 270-272), extensive research data have shown that blood alcohol level  $b$  (in percent) is related to the risk  $R$  (in percent) of having an automobile accident. They indicated that the rate of change of the risk  $R$  with respect to the blood alcohol level  $b$  can be given by  $dR/db = kR$ . If the risk of accident is 10% without blood alcohol, at what blood alcohol level will the risk of having an accident be 100% when  $k = 21.4$ ?
11. *Chemistry: Radioactive decay.* Plutonium is a metallic chemical element that is found in trace quantities in native uranium ores. Its isotope, plutonium-239, is used in nuclear weapons and as a reactor fuel. Its presence in the environment is of great concern to human and animal health. Its decay rate is very slow: 0.003% per year (Bittinger and Morrel, 1988. p. 277). Suppose 200 grams of plutonium is present at  $t=0$ , how much will remain after 50 years? What is its half-life?
12. *Chemistry: Carbon Dating.* Carbon dating is the process of establishing the approximate age carbonaceous materials, such as fossil remains and archaeological specimens, by measuring the amount of radioactive carbon-14 remaining in these materials. The radioactive carbon-14 is present in all carbon-containing matter and has a half-life of 5730 years. If the linen wrappings from one of the Dead Sea Scrolls were found to have lost 22.3% of its carbon-14 (Bittinger and Morrel, 1988. p. 279), how old are the linen wrappings?
13. *Chemistry: Newton's law of cooling.* According to the Newton's law of cooling, the temperature  $T$  of a cooling object decreases at a rate proportional to the difference between  $T$  and the surrounding constant temperature  $C$  (Bittinger and Morrel, 1988). The rate of change of  $T$  with respect to  $t$  is given as  $dT/dt = -r(T-C)$ . The solution to the differential equation is  $T(t) = e^{-rt} + C$ . Here is an exercise to apply Newton's law of cooling. A murder was committed (adapted from Bittinger and Morrel, 1988. p. 282-283). The police was called. The body of a dead person was found sprawled on the floor. In order to solve the crime, the time of the murder must be established. The police called the coroner, who arrived at the murder scene at 12:00 noon. He immediately took the temperature of the body and found it to be 94.6°F. He took the temperature

again at 1 p.m. and found it to be 93.4°F. The room temperature was 70°F. Normal body temperature is given as 98.6°F. Find out when the murder was committed.

14. *Environmental: Air pollution.* When a beam of light, with initial intensity  $I_0$ , enters a medium, such as water or air, its intensity is decreased (Bittinger and Morrel, 1988, p. 285). The reduction in intensity is a function of the thickness or concentration of the medium. The intensity  $I$  at a depth or concentration of  $x$  units is given by  $I = I_0 e^{-\mu x}$ , where  $\mu$  is the coefficient of absorption. The value of  $\mu$  varies with the medium. The equation is known as the Beer-Lambert Law. Here is an exercise to apply the Beer-Lambert law. Air pollution is an international concern. In many cities around the world, smog is getting from bad to worse. Air pollution affects the transmission of sunlight into the earth. In a smoggy area,  $\mu$  can have a value of 0.001 (Bittinger and Morrel, 1988, p. 285). If the air in a smoggy city was found to have 100 micrograms per cubic meter of particulate (pollution concentration in the medium; this represents  $x$  in the Beer-Lambert equation), what percentage of an initial amount  $I_0$  of sunlight passed through the smoggy air?

Table 3.6. Worksheet for the systems methodology.

Stage 1: Problem Definition: What is the problem?

Stage 2: Goal Setting: What is your objective? State in measurable terms. Identify constraints, if necessary.

Stage 3: Systems Synthesis: How do you plan to achieve your objective? Identify all possible strategies or actions that will help achieve your objective.

Stage 4: Systems Evaluation: For each strategy enumerated in Stage 3, identify and evaluate adverse consequences that might make it inappropriate or cause it to fail during actual implementation. Try to rate each consequence as to its likelihood of occurrence (1 to 5, where 1 = not likely to occur, 5 = will likely occur) and its seriousness (1 to 5, where 1= not bad, 5 = very bad). A rating sheet is provided below (use additional sheets if necessary):

<u>Strategy/Consequences</u>	<u>Likelihood</u>	x	<u>Seriousness</u>	=
<u>Score</u>				

Stage 5: System Selection: Based on your systems evaluation, which strategy is likely the best solution to the problem? You may now re-rank the items as a team.

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Appendix

**SLCA Project Worksheet**

**Project Title:** \_\_\_\_\_

**Project Team Members:** \_\_\_\_\_

**Phase 1. Goal and Scope Definition**

<b>Section</b>	<b>Description</b>	<b>Project Detail</b>
Goal	Intended application	
	Intended audience	
	Intended use of results	
	Users of results	
	Summary of goal statement	
Flowchart		(attach to document)
Scope	Functions of the system	
	Functional unit	
	System boundaries	
	Allocation procedure	
	Impact assessment	
	Data requirement	
	Assumptions	
	Constraints	

## Phase 2. Inventory Analysis

<b>Section</b>	<b>Description</b>	<b>Project Detail</b>
Life cycle stages	Pre-manufacturing	
	Manufacturing	
	Product delivery	
	Customer use & reuse	
	Recycling & waste management	
	Disposal	
Data	Collection	
	Validation	
	Allocation	

### Phase 3. Impact Analysis

<b>Section</b>	<b>Description</b>	<b>Project Detail</b>
Stressors	Raw materials	
	Energy use	
	Air emissions	
	Liquid discharge	
	Solid waste	
	Radiation	
	Noise	
Impact Categories	Human health	
	Ecological health	
	Social welfare	
	Resource depletion	

**Impact-analysis weighting matrix**

Life Stage	Stressor	Impact Category				Row Average
		Human Health	Ecological Health	Social Welfare	Resource Depletion	
Pre-manufacture						
Product Manufacture						
Product Delivery						
Product use						
Recycling						
Disposal						

#### **Phase 4. Interpretation**

<b>Section</b>	<b>Description</b>	<b>Project Detail</b>
Environmental issues		
Evaluate results	Completeness	
	Sensitivity	
	Consistency	
Conclusion	Consistency with goals	
Recommended Improvement	Product design	
	Product function	
	Product disposal	



## Growth and Feedback in Population Biology

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### PRINCIPLES

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Populations grow and decline and approach their carrying capacities. Competing species exclude one another; and predator-prey systems oscillate.

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**G**rowth and decay are basic principles operating in living organisms. This chapter will present mathematical models that will illustrate the principles of population increase and decline. The model assumptions do not include many of the important environmental factors, stochastic events, and spatial effects, nevertheless, the solutions provide intuition into the dynamics of population species. The importance of these models does not come from the accuracy of the prediction but rather from the fundamental principles that they set forth, such as:

1. populations grow and decline;
2. population growth approaches the carrying capacity;
3. competing species exclude one another;
4. predator-prey systems oscillate; and
5. epidemics are threshold dependent on population size.

TOPICS

- Exponential Growth Equation
- Logistic Equation
- Lotka-Volterra's Predator-Prey Equations
- Multispecies Extension of the Lotka-Voltra Model
- The dynamics of Infection
- Feedback Analysis
- Steady State and the Isocline Analysis
- The SIRS Model

## 4.1 Exponential Growth Equation

A model for single species was first demonstrated by Malthus (1798) and has since become known as the exponential growth equation, in the form:

$$\frac{dP}{dt} = rP, \tag{4.1}$$

where  $r$  is the net growth rate. When  $r > 0$ , the rate of population growth is positive and the population increases over time (Fig. 4.1); when  $r < 0$ , the rate of population growth is negative and the population decreases with time (Fig. 4.2). The solution form of equation (4.1), as derived from the previous chapter, is as follows:

$$P(t) = P_0 e^{rt} \tag{4.2}$$

where  $P_0$  is the initial population. Equation (4.1) reveals that for an isolated population without migration, the rate of growth depends on the population density. The growth curve of equation (4.1) is illustrated in Fig. 4.1.

```
graph1=Plot[Evaluate[ P[t] /. NDSolve [{P'[t]==0.08 P[t], P[0]==10}, P[t], {t,0,80} ]],
{t,0,80}, PlotRange->{{0,80},{0,600}}, AxesLabel-> {"Time","Population"}]
```

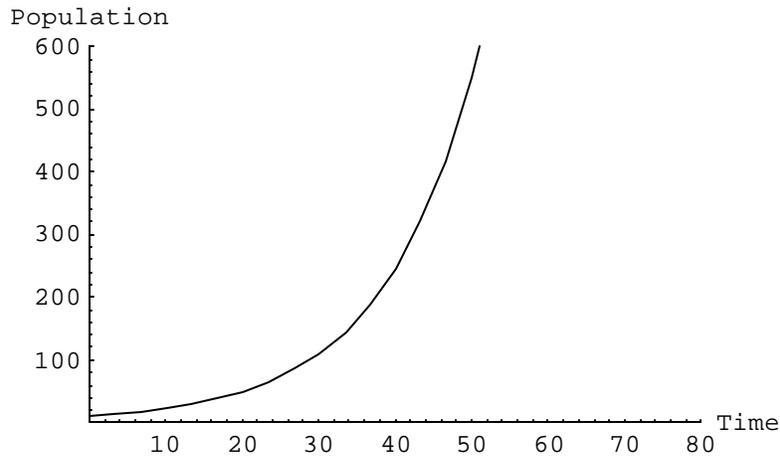


Figure 4.1. A graphical illustration of the exponential growth equation with  $r = 0.08$  and  $P_0 = 10$  over time.

```
graph2=Plot[Evaluate[ P[t] /. NDSolve [{P'[t]==-0.08 P[t], P[0]==600}, P[t], {t,0,80} ]],
{t,0,80}, PlotRange->{{0,80},{0,600}}, AxesLabel-> {"Time","Population"}]
```

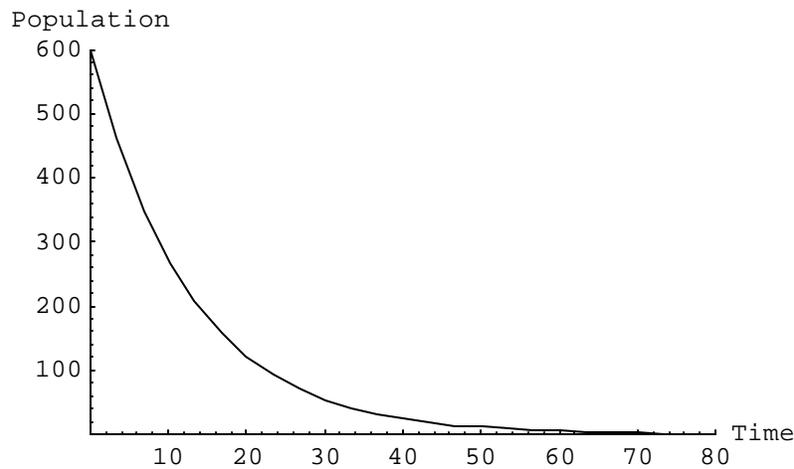


Figure 4.2. A graphical illustration of the exponential growth equation with  $r = 0.08$  and  $P_0 = 10$  over time.

## 4.2 Logistic Equation

In real-life situations, populations do not increase without bounds. When population density becomes high, competition for food, space, and other natural resources become severe. Under crowded conditions, mortality increases. Birth and survival rates drop and the species population approaches the carrying capacity of the environment. The logistic equation, originally derived by Verhulst (1838) and also known as the Verhulst equation, is used to model this phenomenon and written as follows:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{L}\right) \quad (4.3)$$

where  $r$  is the specific growth rate and  $L$  is the carrying capacity of the environment for the species. By simplifying equation (4.3), we get

$$\frac{dP}{dt} = rP - \frac{r}{L}P^2 \quad (4.4)$$

Indeed, we can see that mortality from the crowding effect is pronounced when there are frequent *encounters* between individuals, as represented in equation (4.4) by  $P^2$ . Equation (4.3) shows that when  $P < L$ , then  $\frac{dP}{dt} > 0$  and the population increases with time and approaches the carrying capacity, such as shown in Fig. 4.3. When  $P > L$ ,  $\frac{dP}{dt} < 0$  and the population decreases with time, as shown in Fig. 4.4. Figure 4.5 illustrates the two curves together.

```
graph2=Plot[Evaluate[ P[t] /. NDSolve [{P'[t]==0.08 P[t] (1-P[t]/500), P[0]==10}, P[t],
{t,0,100}]], {t,0,100}, PlotRange->{{0,100},{0,800}}, AxesLabel->
{"Time","Population"}]
```

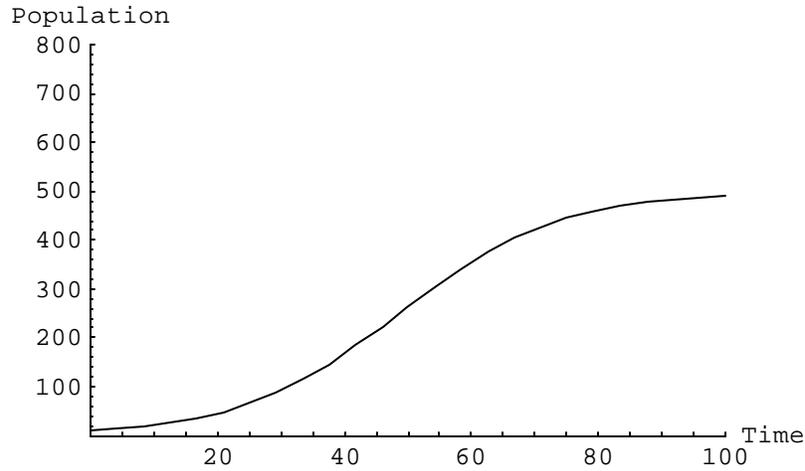


Figure 4.3. Logistic curve, when  $r=0.08$ ,  $L=500$ , and  $P_0=10$ .

```
graph3=Plot[Evaluate[ P[t] /. NDSolve [{P'[t]==0.08 P[t] (1-P[t]/500), P[0]==800}, P[t],
{t,0,100}]], {t,0,100}, PlotRange->{{0,100},{0,800}}, AxesLabel-> {"Time","Population"}]
```

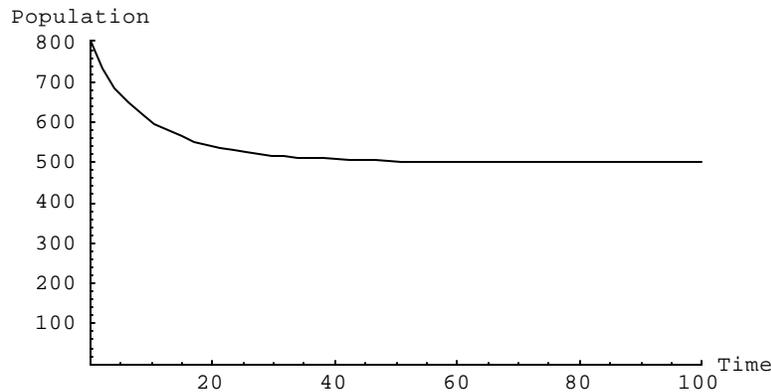


Figure 4.4. Logistic curve, when  $r=0.08$ ,  $L=500$ , and  $P_0=800$ .

```
Show[graph2, graph3]
```

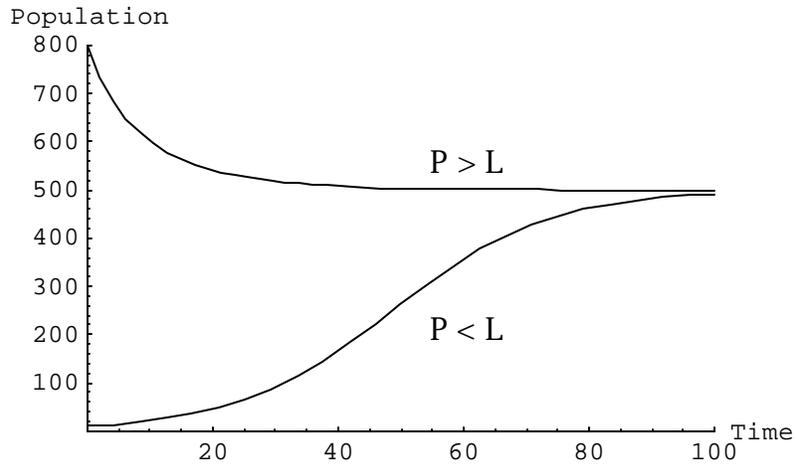


Figure 4.5. Logistic curves, when  $r=0.08$  and  $L=500$  at two initial populations: 10 and 800.

The point at which the rate of change of the population growth shifts from an increasing to a decreasing function is called the inflection point. A graph between population and the rate of change of population  $\frac{dP}{dt}$  is shown in Fig. 4.6.

```
r=0.8;      L=500;
Pdot = r*P (1 -P/L);
graph4=Plot [Pdot, {P,0,500}, PlotRange -> {{0,500},{0,120}}, AxesLabel ->
{"Population", "rate of change"}]
```

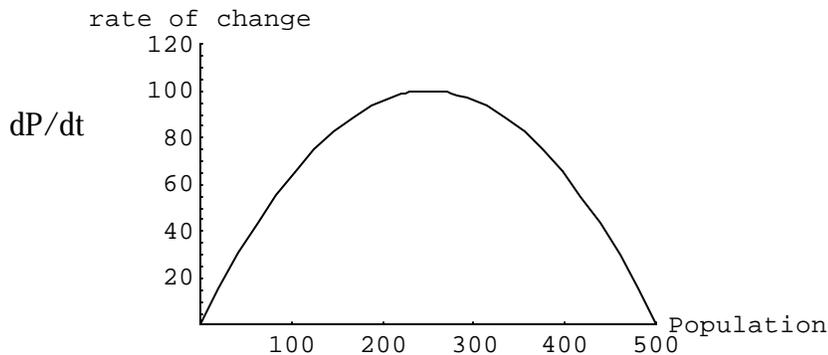


Figure 4.6.  $\frac{dP}{dt}$  curve as a function of the population  $P(t)$ .

We can see from Fig. 4.5 that the rate of change  $\frac{dP}{dt}$  reaches a maximum when  $P=L/2$ . The point,  $P=L/2$ , is called the inflection point.

The analytical solution to equation (4.3) is:

$$P(t) = \frac{P_0 L}{P_0 + (L - P_0)e^{-rt}} \quad (4.5)$$

where  $P_0$  is the initial population.

The exponential and logistic equations are extensions of the infinite power or Taylor series (Edelstein-Keshet, 1988) of the form:

$$f(P) = \sum_{n=0}^{\infty} a_n P^n = a_0 + a_1 P + a_2 P^2 + a_3 P^3 + \dots \quad (4.6)$$

where  $a_0 = 0$ . For the exponential growth equation (4.1),  $\frac{dP}{dt} = f(P)$  and

$f(P) = \sum_{n=0}^1 a_n P^n = a_0 + a_1 P$  where  $a_0=0$  and  $a_1=r$ . For the logistic growth equation (4.3), the function  $f$  is the polynomial

$$f(P) = a_0 + a_1 P + a_2 P^2$$

where  $a_0 = 0$ ,  $a_1 = r$  and  $a_2 = -r/L$ . Thus any growth function may be written as a (possibly infinite) polynomial (Edelstein-Keshet, 1988) of the form

$$\begin{aligned} \frac{dP}{dt} &= a_1 P + a_2 P^2 + a_3 P^3 + \dots \\ &= P(a_1 + a_2 P + a_3 P^2 + \dots), \\ &= P g(P). \end{aligned}$$

where the polynomial  $g(P)$  is called the *intrinsic growth rate* of the population. For an exponential growth equation,  $g(P) = r$  whereas for the logistic equation,

$$g(P) = r \left( 1 - \frac{P}{L} \right).$$

**Example 4.1.** Population dynamics of *Microtus voles*

Graphically show the population dynamics of *Microtus voles* (a boreal rodent), over 2 years if the specific growth rate  $r$  is 5.4 voles  $\text{yr}^{-1}$ , the carrying capacity  $L$  is 100 voles  $\text{ha}^{-1}$ , and the initial population at time zero is 4 (Hanski et al., 1993). Show the population dynamics when the initial population is 150. Identify the inflection point.

```

r=5.4;
L=100;
eq1=P'[t] == r*P[t] (1-P[t]/L);
eqfunc[Pinit_]:=NDSolve[{eq1, P[0]==Pinit}, P[t], {t,0,2}];
initialpop={4,150};
sol=Table[ eqfunc[ initialpop[[j]] ] [[1,1,2]], {j,1,2}];
p1=Plot [ Evaluate [sol], {t,0,2}, PlotRange -> {0,200}, AxesLabel -> {"Years", "Vole
population"} ]

```

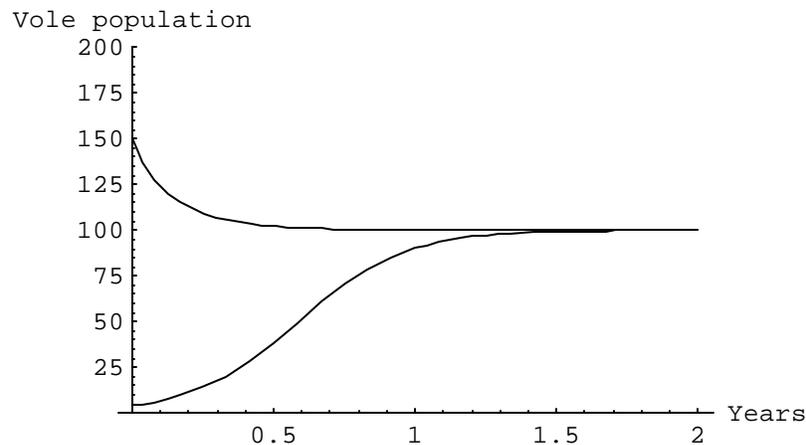


Figure 4.7. Microtus vole population over two years at two initial conditions.

Aside from Gause's work on yeast cultures, such models have been applied to a variety of populations including humans (Pearl and Reed, 1920), microorganisms (Slobodkin, 1954), and other species (Edelstein-Keshet, 1988).

These models can now be used to analyze and answer the following questions: what parameters are affected when environmental pollution occurs? When there is lack of food? And so on.

### 4.3 Lotka-Volterra's Predator-Prey Equations

Other than environmental resistance, growth processes are also affected by interaction among populations of species. The relationship can be described as neutralism, mutualism or symbiosis, synergism, commensalism, amensalism, competition, and predation. These phenomena are observed among chemical compounds, biological species, political parties, business enterprises, and many more.

Neutralism represents a lack of interaction between two populations. The significance of neutralism is difficult to evaluate. It can occur if populations are far apart. It would be found most frequently in very dilute populations where competition would be minimal or when the organisms use different substrates and need not compete. Neutralism has been observed in a chemostat culture of *Lactobacillus* and *Streptococcus* yoghurt starter cultures. The individual population

sizes in the chemostat were found to be the same as the separate mono-cultures under the same conditions.

Synergism is a form of interaction where organisms benefit from the relationship but the association is not obligatory.

Mutualism is obligatory relationship between two populations that benefits both populations. In mutualism, the interaction is necessary for survival. *Symbiosis* describes specific interactions that cannot be performed alone, and like synergism, are not necessary for survival. One case of mutualism is the action between *L. arabinosus* and *S. faecalis* which release phenylalanine and folic acid respectively. There are numerous cases of symbiosis involving plant and animals and microbes, e.g., the interaction of insects with yeast.

Amensalism is association which is detrimental to one species and neutral to the other. In amensalism, the unaffected population releases agents such as antibiotics that restrict the growth of the other population. Amensalistic interactions can induce dormancy or morphogenesis in the antagonized population. One case with relevance to the milk fermentation process is the interaction between the nisin-producing *S. lactis* and the restricted *L. casei*. Non-specific amensalism can result from the lowering of the system pH or the production of oxygen or hydrogen peroxide. There is some overlap between amensalism and competition, although the latter must be a two-way process.

Commensalism is association in which one organism is benefited and the other organism is neither benefited nor harmed. Commensalism is common and is a significant factor in organism succession within microbial communities. An example is *S. cerevisiae* releasing riboflavin for use by *L. casei* for growth. Prior modification of the environment by one organism often leads to better growth for another organism. Examples of environmental changes include the removal of oxygen for anaerobic processes, raising the water content, reducing the osmotic pressure and detoxifying the environment. This process is called metabiosis and is used for the production of the Japanese drink sake.

Parasitism is when the predator is smaller than the prey. A wide range of microbial groups contain parasitic members: viruses, bacteria, and amoebae.

Symbiosis means coexistence or living together; it is not specific enough to deserve an entry in the table. In complicated mixtures of organisms, individuals usually play several roles. For example, they can prey on others while being themselves the food for larger organisms.

Competition is an interactive association between two species both of which need some limited environmental factor for growth and thus grow at suboptimal rates because they must share the growth limiting resource. Competition is the single most important interaction in nature. Experiments have shown that some or all of the less competitive organisms are not eliminated, but

stable coexistence can occur. Commensalism and mutualism among some of the organisms could be responsible for this, or it may result from a population that preys on the competing organisms. The competing populations can be stabilized by feedback control.

Predation is an interaction between organisms in which one benefits and one is harmed based on the ingestion of the smaller sized organism, the prey, by the larger organism, the predator. Predation is brought about by endocytosis and is restricted to certain protozoa and others with amoebal phases. The end result is an oscillation between the predator and the prey.

Interaction	Species 1	Species 2
Neutralism	0	0
Mutualism	+	+
Competition	-	-
Commensalism	0 (or +)	+ (or 0)
Amensalism	0 (or -)	- (or 0)
Predation	+ (or -)	- (or +)

Two species living close to one another usually compete for the resources, habitat, or territory. According to the *principle of competitive exclusion*, the strongest prevails, driving the weaker competitor to extinction. The species whose members are more efficient at finding or exploiting resources wins and leads to an increase in its population. Growth of two competing species in a closed environment has been analyzed by Vito Volterra, a famous Italian mathematician (Bailey and Ollis, 1986). Volterra explained the oscillations in fish populations in the Mediterranean by proposing the following reasoning (Edelstein-Keshet, 1988):

1. Unlimited growth of prey in the absence of predators;
2. Predator survival depends on the presence of their prey;
3. The rate of predation depends on the likelihood that a victim is encountered by a predator; and
4. The growth rate of the predator population is proportional to food intake (rate of predation).

With these assumptions, Volterra formulated the following equations:

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy, \\ \frac{dy}{dt} &= -cy + fxy, \end{aligned} \tag{4.7}$$

where  $x$  and  $y$  represent prey and predator populations respectively; and the variables  $a$ ,  $b$ ,  $c$ , and  $f$  may represent biomass or population densities of the species. Equation 4.7 is also known as the Lotka-Volterra predator-prey model developed by Lotka and Volterra in the late 1920a (Bailey and Ollis, 1986).

When  $y=0$  (no predators), the net growth rate  $a$  of the prey population is a positive quantity in accordance with assumption 1. When  $x=0$  (no prey), predators die (the net death rate  $c$  is positive valued) following assumption 2. The term  $xy$  approximates the likelihood of encounter between predators and prey given that both species move about randomly and are uniformly distributed over their habitat. The form of this encounter rate is derived from the *law of mass action* that states that the rate of molecular collisions of two chemical species in a dilute gas or solution is proportional to the product of the two concentrations (Edelstein-Keshet, 1988). An encounter between  $x$  and  $y$  is assumed to decrease the prey population due to predation and increase the predator population due to food intake. The ratio  $b/f$  is the number of prey per predator, analogous to the efficiency of predation. Equation (4.7) is demonstrated in Fig. 4.8 for a case as follows: the net growth rate of the prey ( $a$ ) is 0.8 per year, the predation factor  $b$  is 0.05 per predator per year, the death rate of the predator is 0.8 per year ( $c$ ), and the predation factor  $f$  is 0.8 per prey per year.

```

a = 0.8; b = 0.05; c = 0.8; f = b*0.8;
sol = NDSolve [{x'[t] == a x[t] - b x[t] y[t], y'[t] == -c y[t] + f x[t] y[t], x[0]==20, y[0]==10},
{x,y}, {t,0,60}];
plot1=Plot [Evaluate [ {x[t], y[t]} /. sol], {t,0,30}, PlotRange-> {{0,30},{0,40}}, Compiled->
False, AxesLabel->{"Years", "Populations"}]

```

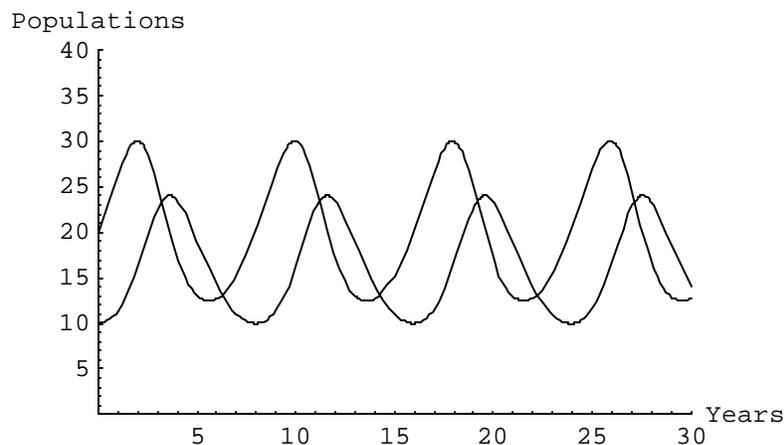


Figure 4.8. Predator-prey population dynamics.

Example 4.2. Population dynamics of the *Microtus voles* and the least weasels

Consider two species where one is the food source of the other (Beltrami, 1987). Let the prey population be represented as  $P_1$  and the predator population be  $P_2$ . In this situation,  $P_1$  is the only food source of  $P_2$ , so that when  $P_1$  is abundant,  $P_2$  is able to increase in its population, but in the absence of  $P_1$ ,  $P_2$  dies out at a constant per-capita rate of  $c > 0$ . When predation is absent,  $P_1$  is assumed to grow logistically. In the presence of mutual interaction,  $P_1$  population decreases while  $P_2$

population increases. The interaction between the two species is assumed to occur in proportion to the total number of possible ways in which  $P_1$  and  $P_2$  meet. This concept is mathematically modeled as the product of the two populations, that is,  $P_1P_2$ . The predator-prey model assumes no outside interference and is given as follows:

$$\frac{dP_1}{dt} = rP_1 \left( 1 - \frac{P_1}{L} \right) - aP_1P_2$$

$$\frac{dP_2}{dt} = -cP_2 + bP_1P_2$$

Let the *Microtus* voles be the prey (and its population  $P_1$ ) and the least weasel (mustelids) be the predator (and its population  $P_2$ ). Let the following parameters be defined: the annual growth rate of  $P_1$  is 5.4 voles  $\text{yr}^{-1}$  ( $r$ ); the carrying capacity of  $P_1$  is 100  $\text{ha}^{-1}$  ( $L$ ); the variable  $a$  is the consumption factor of the predator ( $P_2$ ) on the prey ( $P_1$ ), where  $a=4$ ; the variable  $b$  is the per-capita growth rate of  $P_2$  resulting from its catch of  $P_1$ , where  $b=10\%$  of  $a$ ; and  $c = 2.5$ , is the per-capita death rate of  $P_2$ . The variable  $c$  can also mean to be the harvesting rate of  $P_2$ . Estimate the population over 10 years when the initial populations are 4 for  $P_1$  and 2 for  $P_2$ .

```
r=5.4; L=100; a=4; c=2.5; b=0.1*a;
prey = P1'[t] == r*P1[t] (1 - P1[t]/L) - a*P1[t]*P2[t];
pred = P2'[t] == - c*P2[t] + b*P1[t]*P2[t];
eq1 =NDSolve[{prey,pred, P1[0]==4, P2[0]==2}, {P1[t], P2[t]}, {t,0,10}];
p1=Plot [ Evaluate [{P1[t],P2[t]} /. eq1], {t,0,10},
  AxesLabel -> {"Years", "Voles and weasel populations"} ]
```

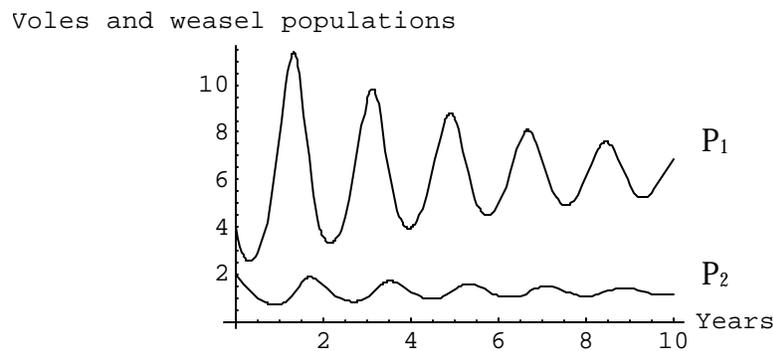


Figure 4.9. Population dynamics of the voles ad weasels.

The Lotka-Volterra model for species competition is given in the following form:

$$\frac{dP_1}{dt} = r_1 P_1 \left( 1 - \frac{P_1}{L_1} \right) - \frac{r_1 \mathbf{b}_{12}}{L_1} P_1 P_2 \quad (4.9a)$$

$$\frac{dP_2}{dt} = r_2 P_2 \left( 1 - \frac{P_2}{L_2} \right) - \frac{r_2 \mathbf{b}_{21}}{L_2} P_1 P_2 \quad (4.9b)$$

where  $P_1$  and  $P_2$  are the population densities of species 1 and 2. The equations are developed based on several assumptions, namely:

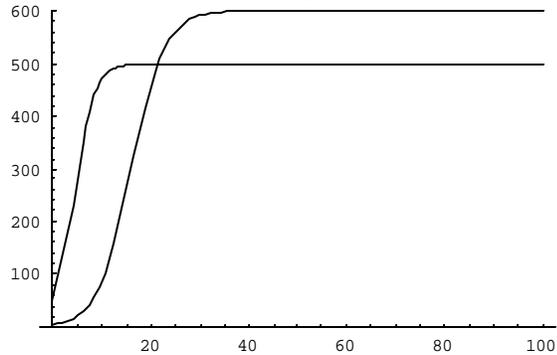
1. In the absence of a competitor ( $P_2=0$  for equation (4.9a) and  $P_1=0$  for equation (4.9b), the two equations reduce to the logistic equation (4.3). The population of species 1 will stabilize at the value  $P_1=L_1$  and the population of species 2 will stabilize at the value  $P_2=L_2$ .
2. The parameters  $\mathbf{b}_{12}$  and  $\mathbf{b}_{21}$  are the per capita decline caused by individuals of species 2 (species 1) on the population of species 1 (species 2).

Some interactions are illustrated mathematically below.

```

r1 = 0.5;
r2 = 0.3;
L1 = 500;
L2 = 600;
a1 = 0.2;
a2 = 0.3;
H*Neuralism*L
NDSolve@8p1 '@tD Š r1 p1@tD H1 - p1@tD • L1L,
  p2 '@tD Š r2 p2@tD H1 - p2@tD • L2L,
  p1@0D Š 50, p2@0D Š 5<, 8p1@tD, p2@tD<, 8t, 0, 100< D
Plot@Evaluate @8p1@tD, p2@tD< • . %D, 8t, 0, 100<,
  PlotRange @ 80, 601<D
88p1@tD @ InterpolatingFunction@880., 100.<<, <>D@tD,
  p2@tD @ InterpolatingFunction@880., 100.<<, <>D@tD<<

```



... Graphics ...

### H\*Mutualism\*L

NDSolve@

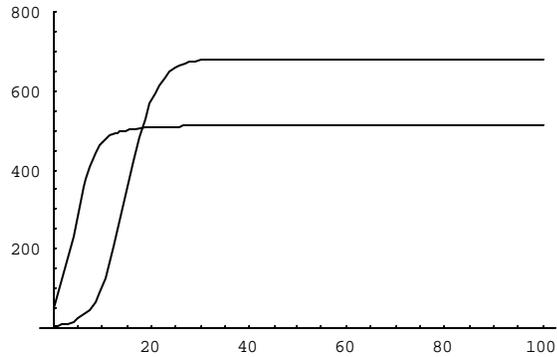
```
8p1 '@tD Š r1 p1@tD H1 - p1@tD • L1L + 0.00002 * p1@tD p2@tD,  
p2 '@tD Š r2 p2@tD H1 - p2@tD • L2L + 0.00008 * p1@tD p2@tD,  
p1@0D Š 50, p2@0D Š 5<, 8p1@tD, p2@tD<, 8t, 0, 100<D
```

Plot@Evaluate@8p1@tD, p2@tD< •. %D, 8t, 0, 100<,

PlotRange @ 80, 801<D

88p1@tD @ InterpolatingFunction@880., 100.<<, <>D@tD,

p2@tD @ InterpolatingFunction@880., 100.<<, <>D@tD<<



... Graphics ...

### H\*Predation\*L

q1 =

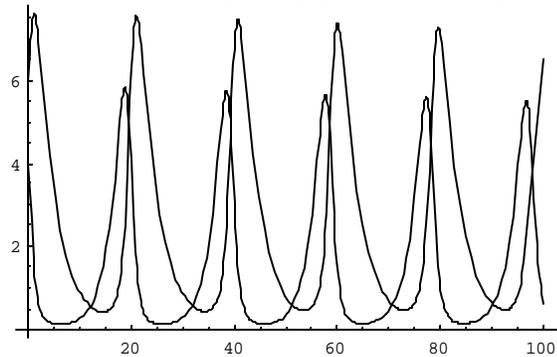
NDSolve@

```
8p1 '@tD Š r1 p1@tD H1 - p1@tD • L1L - a1 * p1@tD p2@tD,  
p2 '@tD Š - a2 * p2@tD + a1 * p1@tD p2@tD,  
p1@0D Š 4, p2@0D Š 6<, 8p1@tD, p2@tD<, 8t, 0, 100<D
```

88p1@tD @ InterpolatingFunction@880., 100.<<, <>D@tD,

p2@tD @ InterpolatingFunction@880., 100.<<, <>D@tD<<

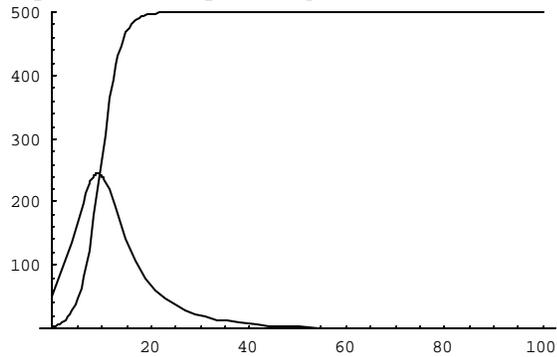
Plot@Evaluate@8p1@tD, p2@tD< •. q1D, 8t, 0, 100<D



... Graphics ...

H\*Amensalism\*L

```
q2 = NDSolve@8p1 '@tD Š r1 p1@tD H1 - p1@tD • L1L,  
      p2 '@tD Š r2 p2@tD H1 - p2@tD • L2L - 0.0008 * p1@tD p2@tD,  
      p1@0D Š 4, p2@0D Š 50<, 8p1@tD, p2@tD<, 8t, 0, 100<D  
Plot@Evaluate@8p1@tD, p2@tD< •. q2D, 8t, 0, 100<,  
      PlotRange@ 80, 501<D  
88p1@tD® InterpolatingFunction@880., 100.<<, <>D@tD,  
p2@tD® InterpolatingFunction@880., 100.<<, <>D@tD<<
```

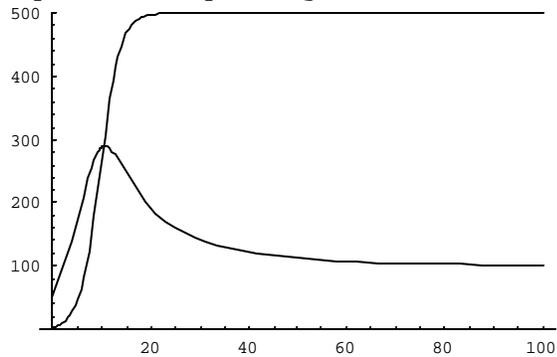


...Graphics ...

H\*Amensalism - negative effect is 0.0005 ®

p2 survives\*L

```
q3 = NDSolve@8p1 '@tD Š r1 p1@tD H1 - p1@tD • L1L,  
      p2 '@tD Š r2 p2@tD H1 - p2@tD • L2L - 0.0005 * p1@tD p2@tD,  
      p1@0D Š 4, p2@0D Š 50<, 8p1@tD, p2@tD<, 8t, 0, 100<D  
Plot@Evaluate@8p1@tD, p2@tD< •. q3D, 8t, 0, 100<,  
      PlotRange@ 80, 501<D  
88p1@tD® InterpolatingFunction@880., 100.<<, <>D@tD,  
p2@tD® InterpolatingFunction@880., 100.<<, <>D@tD<<
```



...Graphics ...

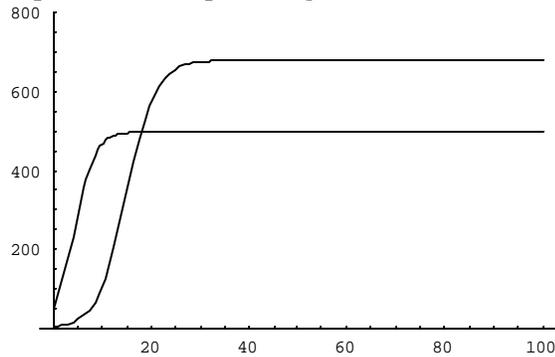
H\*Commensalism\*L

```
q4 = NDSolve@8p1 '@tD Š r1 p1@tD H1 - p1@tD • L1L,  
      p2 '@tD Š r2 p2@tD H1 - p2@tD • L2L + 0.00008 * p1@tD p2@tD,  
      p1@0D Š 50, p2@0D Š 5<, 8p1@tD, p2@tD<, 8t, 0, 100<D  
Plot@Evaluate@8p1@tD, p2@tD< •. q4D, 8t, 0, 100<,  
      PlotRange@ 80, 801<D
```

```

88p1@tD @ InterpolatingFunction@880., 100.<<, <>D@tD,
p2@tD @ InterpolatingFunction@880., 100.<<, <>D@tD<<

```



...Graphics ...

**H\*Competition\*L**

q5 =

**NDSolve@**

```

8p1 '@tD Š r1 p1@tD H1 - p1@tD • L1L - 0.0002 * p1@tD p2@tD,
p2 '@tD Š r2 p2@tD H1 - p2@tD • L2L -
0.00008 * p1@tD p2@tD,
p1@0D Š 50, p2@0D Š 5<, 8p1@tD, p2@tD<, 8t, 0, 100< D

```

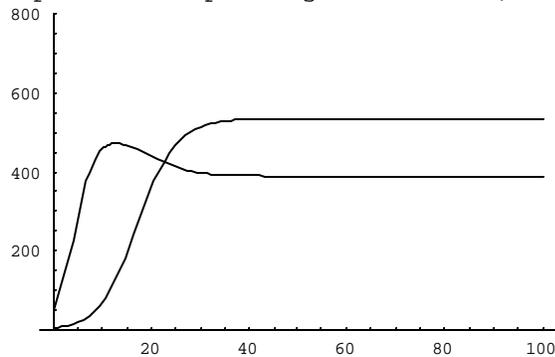
**Plot@Evaluate @8p1@tD, p2@tD< • . q5D, 8t, 0, 100<,**

**PlotRange @ 80, 801<D**

```

88p1@tD @ InterpolatingFunction@880., 100.<<, <>D@tD,
p2@tD @ InterpolatingFunction@880., 100.<<, <>D@tD<<

```



...Graphics ...

**H\*Competition\*L**

**NDSolve@**

```

8p1 '@tD Š r1 p1@tD H1 - p1@tD • L1L - 0.0009 * p1@tD p2@tD,
p2 '@tD Š r2 p2@tD H1 - p2@tD • L2L - 0.00008 * p1@tD p2@tD,
p1@0D Š 50, p2@0D Š 5<, 8p1@tD, p2@tD<, 8t, 0, 100< D

```

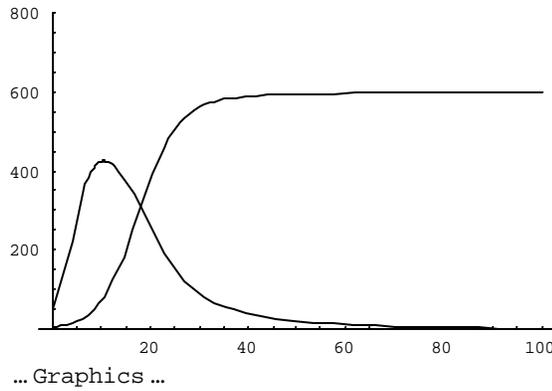
**Plot@Evaluate @8p1@tD, p2@tD< • . %D, 8t, 0, 100<,**

**PlotRange @ 80, 801<D**

```

88p1@tD @ InterpolatingFunction@880., 100.<<, <>D@tD,
p2@tD @ InterpolatingFunction@880., 100.<<, <>D@tD<<

```



#### 4.4 Multispecies Extension of the Lotka-Volterra Model

Ecologists are interested in the relationship between system complexity and its dynamic behavior. For example, is the system more stable or does it exhibit oscillation as the number of different species is increased? What is the effect of more intense interactions? An analogous set of equations for  $N$  predators and  $M$  preys is presented below (Bailey and Ollis, 1986):

$$\frac{dn_i}{dt} = n_i \left( a_i - \sum_{j=1}^N a_{ij} m_j \right) \quad i = 1, \dots, N$$

$$\frac{dm_i}{dt} = m_i \left( -b_i + \sum_{j=1}^M b_{ij} n_j \right) \quad i = 1, \dots, M$$

where  $n_1, \dots, n_N$  are the numbers in the prey populations and  $m_1, \dots, m_M$  are the predator population numbers.

#### 4.5 The Dynamics of Infection

This section illustrates the importance of understanding the population dynamics of disease infection. The discussion and presentation are adapted from Edelstein-Keshet (1988, pp. 242-256).

Infectious diseases can either be microparasitic (caused by viruses and bacteria) or macroparasitic (due to worms). Microparasites reproduce within their host and are transmitted directly from one host to another while macroparasites have more complicated life cycles, often with a secondary host or carrier.

In this exercise, we shall model the process of infection of a viral disease, such as measles or smallpox. Let the population densities of the human host be defined as  $x$  and the viral agent be defined as  $y$ . The assumptions are as follows: (1) the human birth rate  $\alpha$  is constant; (2) viral infection increases mortality, so  $g(y) > 0$ ; (3) reproduction of virus depends on human presence by a proportionality

constant  $\beta$ ; and (4) in the absence of human hosts, virus population “die” or become nonviable at rate  $g$ . The equations are formulated as follows:

$$\begin{aligned}\frac{dx}{dt} &= [\mathbf{a} - g(y)]x, \\ \frac{dy}{dt} &= \mathbf{b}xy - \mathbf{g}y.\end{aligned}\tag{4.10}$$

Equations (4.10) show a modified Volterra’s predator-prey model. We can assume that the virus population  $y$  are predatory organisms searching for human prey  $x$  to consume.

According to Edelstein-Keshet (1988), the philosophical view of disease infection as a process of predation is a misleading analogy for three reasons. First, it is difficult to measure or even estimate total viral population, which may range over several orders of magnitude in individual hosts. Second, a knowledge of this number is trivial since it is the distribution of viruses over hosts that determines what percentage of people will actually suffer from the disease. And third, an underlying hidden assumption in equations (4.10) is that viruses roam freely in the environment, randomly encountering new hosts. This is rarely true of microparasitic diseases. Rather, diseases are spread by contact or close proximity between infected and healthy individuals.

Edelstein-Keshet (1988) suggested that a new approach is necessary: to make a distinction between sick individuals who harbor the disease and those who are as yet healthy. The host population is subdivided into distinct classes according to the health of its members. A typical subdivision consists of susceptibles  $S$ , infectives  $I$ , and a removed class  $R$  of individuals who can no longer contract the disease because they have either recovered with immunity, have been placed in isolation, or have died. If the disease confers a temporary immunity on its victims, individuals can also move from the  $R$  class to the  $S$  class. Time scales of epidemics can vary from days to years. Factors influencing a disease outbreak depend on the vital dynamics of a population (the normal rates of birth and death without the disease) and whether or not immunity is conferred on individuals.

According to Edelstein-Keshet (1988), Kermack and McKendrick (1927) made a major contribution to the theory of epidemics. One of the special cases is illustrated in Fig. 4.10. The diagram summarizes transition rates between the three classes with the parameter  $\beta$ , the rate of transmission of the disease, and the rate of removal  $v$ . It is assumed that each compartment consists of identically healthy or sick individuals and that no births or deaths occur in the population. The situation in Fig. 4.10 is called a SIR model. The rate of transmission of a microparasitic disease is proportional to the rate of encounter of susceptible and infective individuals modelled by the product  $(\beta SI)$ .

The equations due to Kermack and MacKendrick for the disease shown in Fig. 4.10 are (Edelstein-Keshet, 1988):

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \nu I, \quad \frac{dR}{dt} = \nu I. \quad (4.11)$$

The total population  $N$  is the sum of the population of each compartment, that is,  $N=S+I+R$ , where  $N$  is constant.

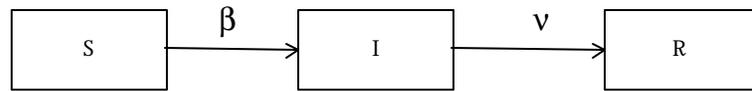


Figure 4.10. Compartment system of the SIR model.

The case in equation (4.11) can be modified to allow for a loss of immunity on the class  $R$  where recovered individuals can become susceptible again, as illustrated in Fig. 4.11. It is assumed that the loss of immunity takes place at a rate proportional to the population in class  $R$ , with proportionality constant  $\gamma$ . Thus the equations become (Edelstein-Keshet, 1988)

$$\frac{dS}{dt} = -\beta SI + \gamma R \quad (4.12a)$$

$$\frac{dI}{dt} = \beta SI - \nu I \quad (4.12b)$$

$$\frac{dR}{dt} = \nu I - \gamma R. \quad (4.12c)$$

The model in equations (4.12) is called a SIRS model since removed individuals can return to class  $S$ .

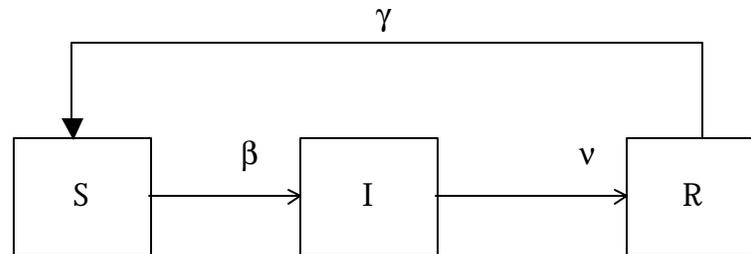


Figure 4.11. Compartment system of the SIRS model.

Models for infectious diseases lead to a better understanding of how vaccination programs affect the control or eradication of the disease and how an immunization can reduce or eliminate the incidence of infection, even when only part of the population receives the treatment. Those individuals who have been vaccinated will acquire protection from the infection. The vaccinated individuals are removed from participating in the transmission of the disease. There will be fewer infectious individuals and thus a decreased likelihood that an unvaccinated susceptible will come in contact with the disease.

**Example 4.3. The spread of infectious disease**

The spread of infectious disease is a function of the rate at which susceptible individuals come in contact with the disease carriers and the probability that those contacts will result in an infection. Factors that increase human contact or diminish a person's ability to resist or recover from the disease has the potential of increasing the likelihood of an epidemic. Population density, climate, lifestyle of the people, social status, sanitary conditions of the community, water and waste management, and economic conditions influence the frequency that individuals come in contact with each other. The ability to resist infection is affected by economic status, stress level, age, health, and hygiene of susceptible individuals and the cost, availability and effectiveness of health care for treatment and prevention.

Suppose an epidemic breaks out. The infectious disease is spread among the population by contact between individuals. Let  $S$  be the fraction of the population that is susceptible to the disease by contact,  $I$  be the fraction of the population that is infectious with a daily recovery rate of  $b > 0$ . Let  $R$  be the fraction of the population that is immune and cannot spread the disease. A model is formulated as follows:

$$\begin{aligned} \frac{dS}{dt} &= -aIS \\ \frac{dI}{dt} &= aIS - bI \\ R &= 1 - S(t) - I(t) \end{aligned} \tag{4.13}$$

where  $a$  is the number of contacts adequate to transmit the disease/infective/day and  $b$  is the daily recovery rate. Let  $a = 0.47$ ;  $b = 0.33$ ;  $S(0) = 0.9999987$ ; and  $I(0) = 0.0000013$ .

```

a=0.47; b=0.33;
eq1=Sus'[t]==-a*Sus[t]*Infec[t];
eq2=Infec'[t]==a*Sus[t]*Infec[t] - b*Infec[t];
eq3=NDSolve[{eq1, eq2, Sus[0]==0.987, Infec[0]==0.013}, {Sus, Infec}, {t,0,200}];
eq4=Rec[t]=1-Sus[t]-Infec[t];
p1=Plot [ Evaluate [{Sus[t], Infec[t]} /. eq3], {t,0,100}, PlotRange->{{0,100},{0,1}},
AxesLabel -> {"Days", "Susceptibles and Infectives"} ]
p2=Plot [ Evaluate [{Rec[t]=1-Sus[t]-Infec[t]} /. eq3], {t,0,100}, PlotRange-
>{{0,100},{0,1}}, AxesLabel -> {"Days", "Recovered"} ]

```

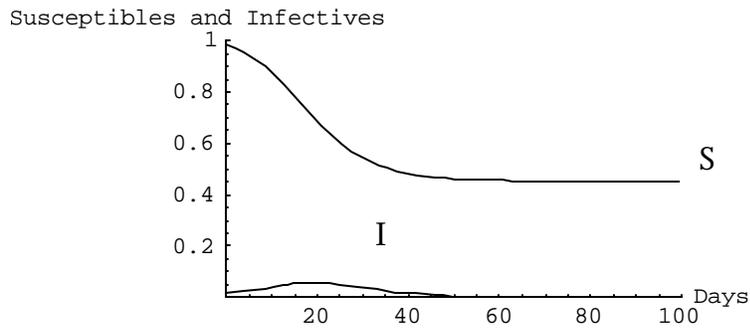


Figure 4.12. Graph showing the interaction between the susceptible and infective populations over time.

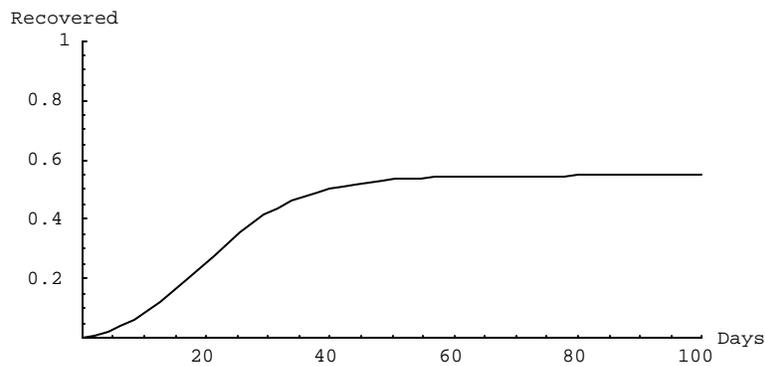


Figure 4.13. Graph showing the recovered population curve over time.

Show [p1, p2]

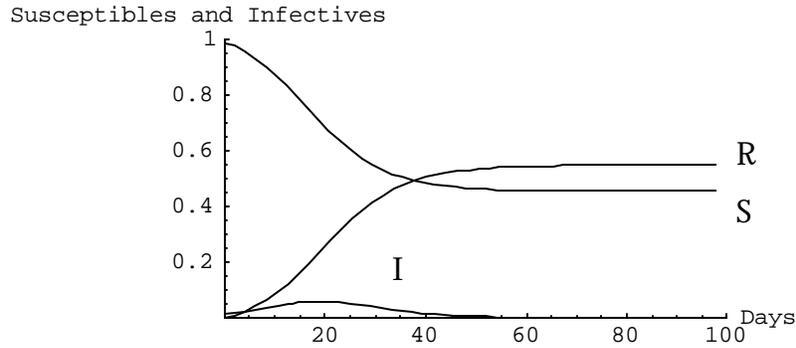


Figure 4.14. Graph showing the dynamics of the susceptible, infective, and recovered populations over time.

Figures 4.12 through 4.14 show that as the infected population (I) increases, the susceptible population (S) decreases and the recovered population (R) increases. When I is at its maximum population, the rate of infection (slope of S vs time) is highest. As the pool of susceptible population (S) decreases and the infectious population decreases, the rate of recovery ( $bI$ ) exceeds the rate of new infections ( $aSI$ ) and I approaches toward zero. At this point the system reaches steady state. This general scenario is true for cases in which  $0 < b < a < 1$  and  $R = 0$ . In such cases, I reaches a characteristic maximum (I max) at some time  $t_{max}$ , indicating the intensity and timing of the epidemic, respectively. R then converges to a maximum value as  $t$  goes to infinity.

#### Feedback Analysis of a Predator-Prey System

Consider a problem where the insects are a pest and the bugs, which feed on the insects, are beneficial to certain useful plants. Pesticide is sprayed to kill or "harvest" the insects, however in the process, the bugs are also killed. Let  $P_1$  and  $P_2$  be the insect and bug populations, respectively. Let  $E_f$  be the per-capita "harvest" rate, equivalent to the efficiency of the pesticide to eliminate both the insects and the bugs. The amount of insects and bugs killed by the pesticide are  $E_f P_1$  and  $E_f P_2$ , respectively. The insect-bug population dynamic is represented as follows:

$$\begin{aligned} \frac{dP_1}{dt} &= rP_1 \left( 1 - \frac{P_1}{L} \right) - aP_1P_2 - E_f P_1 \\ \frac{dP_2}{dt} &= -cP_2 + bP_1P_2 - E_f P_2 \end{aligned} \tag{4.14}$$

The steady-state solutions without pesticide application are:  $P_1=0$ ,  $P_1=\frac{c}{b}$ ;  $P_2=0$ ,  $P_2=\frac{r}{a}\left(1-\frac{c}{bL}\right)$ . What is the steady state solution when pesticide is applied? If  $r=5.4$ ,  $L=100$ ,  $a=4$ ,  $c=2.5$ , and  $b=0.1$ , compare the results with and without pesticide application. Is pesticide application a good practice? Why?

```
r=5.4; L=100; a=4; c=2.5; b=0.1*a; Ef=0.8;
prey = P1'[t] == r*P1[t] (1 - P1[t]/L) - a*P1[t]*P2[t] - Ef*P1[t];
pred = P2'[t] == - c*P2[t] + b*P1[t]*P2[t] - Ef*P2[t];
eq3 =NDSolve[{prey,pred, P1[0]==4, P2[0]==2}, {P1[t], P2[t]}, {t,0,10}];
p3=Plot [ Evaluate [{P1[t],P2[t]} /. eq3], {t,0,10},
  AxesLabel -> {"Years", "Insect and bug populations"} ]
```

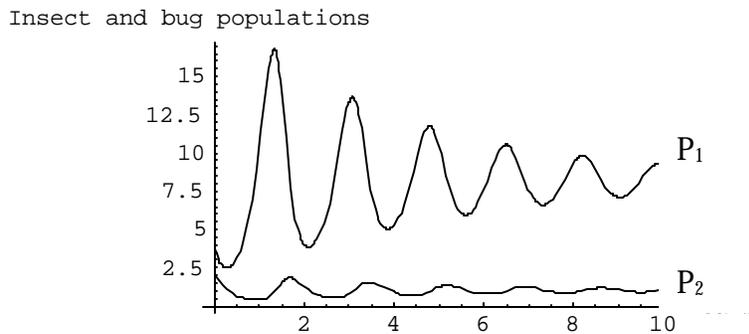


Figure 4.17. Growth curves of the insect and bug populations.

#### 4.6. Steady State and Isocline Analysis

Consider two species competing for the same food source. Let the population of the two species be represented as  $P_1$  and  $P_2$ .  $P_1$  and  $P_2$  would grow logistically in isolation from each other but, when they interact, one interferes with the growth of the other. The interaction between  $P_1$  and  $P_2$  is proportional to the product of their respective populations. The interaction has the effect of reducing the growth rates of  $P_1$  and  $P_2$  as they struggle for the same food source. Let  $L_1$  and  $L_2$  be the carrying capacities of  $P_1$  and  $P_2$ , respectively. The specific growth rates of  $P_1$  and  $P_2$  are  $r_1$  and  $r_2$ , respectively. Let  $a_1$  and  $a_2$  represent the per-capita rates of competitive advantage of one species over the other. When  $a_1 > a_2$ ,  $P_2$  is more efficient as a competitor; and vice versa when  $a_2 > a_1$ . The competition between two species can be modeled with two logistic equations incorporating factors for competition, and written as follows:

$$\begin{aligned}\frac{dP_1}{dt} &= r_1 P_1 \left(1 - \frac{P_1}{L_1}\right) - a_1 P_1 P_2 \\ \frac{dP_2}{dt} &= r_2 P_2 \left(1 - \frac{P_2}{L_2}\right) - a_2 P_1 P_2\end{aligned}\tag{4.15}$$

For two equally competent competitors, coexistence may not be possible. This principle is referred to as the competitive exclusion principle or the Volterra-Gause principle. However, conventional wisdom shows that this is not necessarily true: there are competitors that have co-existed. There are conditions that lead to the mutual existence of competitors, namely:

1. geographical isolation - the species are located far from each other
2. ecological differentiation - species specialize and adapt to their immediate environment
3. interbreeding and mergers - two competing species interbreed or two competing business companies merge.

#### Example 4.4. Populations without competition

Estimate the population of *Paramecium caudatum* ( $P_1$ ) and *Paramecium aurelia* ( $P_2$ ) (Pascual and Kareiva, 1996) **without competition** after 20 days if the growth rate is 0.9 and 0.6  $d^{-1}$ , respectively, and the carrying capacities of *P. caudatum* and *P. aurelia* are 5000 and 2000, respectively. Let the initial population of *P. caudatum* be 200 and that of *P. aurelia* be 100.

```
r1=0.9; L1=5000; r2=0.6; L2=2000;
pop1=P1'[t]==r1*P1[t]*(1-P1[t]/L1);
pop2=P2'[t]==r2*P2[t]*(1-P2[t]/L2);
eq1=NDSolve[{pop1, pop2, P1[0]==200, P2[0]==100}, {P1[t], P2[t]}, {t,0,20}];
p1=Plot[Evaluate[{{P1[t], P2[t]}/. eq1}, {t,0,20}], PlotRange->{0,6000}, AxesLabel->
{"Days", "Populations"}]
```

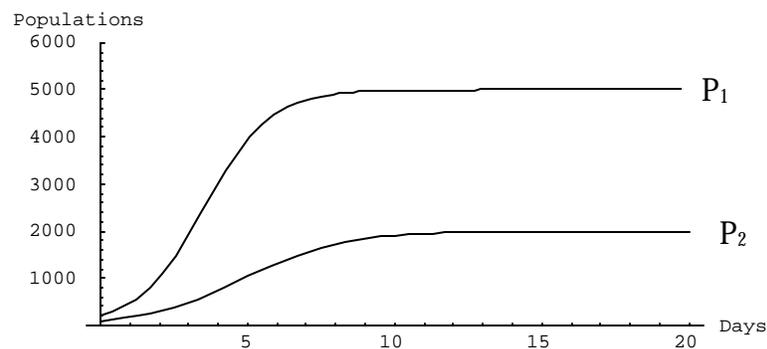


Figure 4.18. Growth curves of *Paramecium caudatum* and *Paramecium aurelia*.

Example 4.5. Populations with competition

Estimate the population of *Paramecium caudatum* and *Paramecium aurelia* (Pascual and Kareiva, 1996) **with competition** after 20 days if the growth rate is 0.9 and 0.6 d<sup>-1</sup>, respectively, and the carrying capacities of *P. caudatum* and *P. aurelia* are 5000 and 2000, respectively. Let the initial population of *P. caudatum* be 200 and that of *P. aurelia* be 100. Let  $P_2$  have a competitive advantage over  $P_1$ .

```
a1=r1/L1 + r1/L1*0.9;
a2=r2/L2 - r2/L2*0.5;
pop3=P1'[t] == r1*P1[t] (1-P1[t]/L1) - a1*P1[t]*P2[t];
pop4=P2'[t] == r2*P2[t] (1-P2[t]/L2) - a2*P1[t]*P2[t];
eq2=NDSolve[{pop3, pop4, P1[0]==200, P2[0]==100}, {P1[t], P2[t]}, {t,0,20}];
p2=Plot[Evaluate[{P1[t],P2[t]} /. eq2], {t,0,20}, PlotRange -> {0,6000}, AxesLabel -> {"Days", "Populations"}]
```

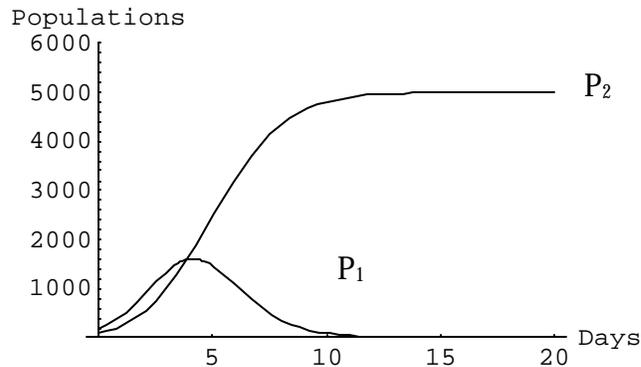


Figure 4.19. Growth curves of *Paramecium caudatum* and *Paramecium aurelia*.

A slight increase in the competitiveness of  $P_2$  causes  $P_2$  population to increase whereas  $P_1$  population goes into extinction.

Example 4.6. Populations with strong competition

Suppose the two competing species have the same growth rates and the same carrying capacities. Let the competitive factors  $a_1$  and  $a_2$  be greater than  $r/L$ . The species with the greater competitive advantage will prevail to the extinction of the other population. This scenario represents strong competition, where  $P_2$  has competitive advantage over  $P_1$ .

```
r1=0.9; L1=5000; r2=r1; L2=L1;
a1=r1/L1 + r1/L1*0.9;
a2=r2/L2 + r2/L2*0.2;
pop3=P1'[t] == r1*P1[t] (1-P1[t]/L1) - a1*P1[t]*P2[t];
pop4=P2'[t] == r2*P2[t] (1-P2[t]/L2) - a2*P1[t]*P2[t];
eq2=NDSolve[{pop3, pop4, P1[0]==200, P2[0]==100}, {P1[t], P2[t]}, {t,0,20}];
p3=Plot[Evaluate[{P1[t],P2[t]} /. eq2], {t,0,20}, PlotRange -> {0,6000}, AxesLabel -> {"Days", "Populations"}]
```

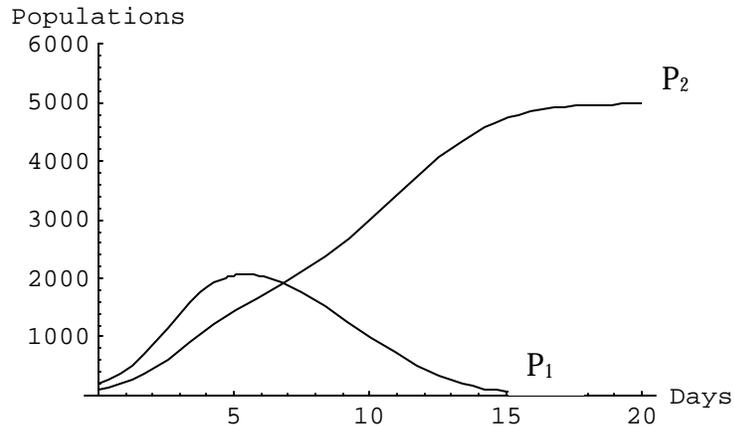


Figure 4.20. Growth curves of *Paramecium caudatum* and *Paramecium aurelia*.

Example 4.7. Populations with mild competition

Suppose the two competing species have the same growth rates and the same carrying capacities. Let the competitive factors  $a_1$  and  $a_2$  be less than  $r/L$ . The species with the greater competitive advantage will have a higher population, however, both species will co-exist. This scenario represents weak competition.

```

r1=0.9; L1=5000; r2=r1; L2=L1;
a1=r1/L1 - r1/L1*0.9;
a2=r2/L2 - r2/L2*0.7;
pop3=P1'[t] == r1*P1[t] (1-P1[t]/L1) - a1*P1[t]*P2[t];
pop4=P2'[t] == r2*P2[t] (1-P2[t]/L2) - a2*P1[t]*P2[t];
eq2=NDSolve[{pop3, pop4, P1[0]==200, P2[0]==100}, {P1[t], P2[t]}, {t,0,20}];
p4=Plot[Evaluate[{{P1[t],P2[t]} /. eq2], {t,0,20}, PlotRange -> {0,6000}, AxesLabel ->
{"Days", "Populations"} ]

```

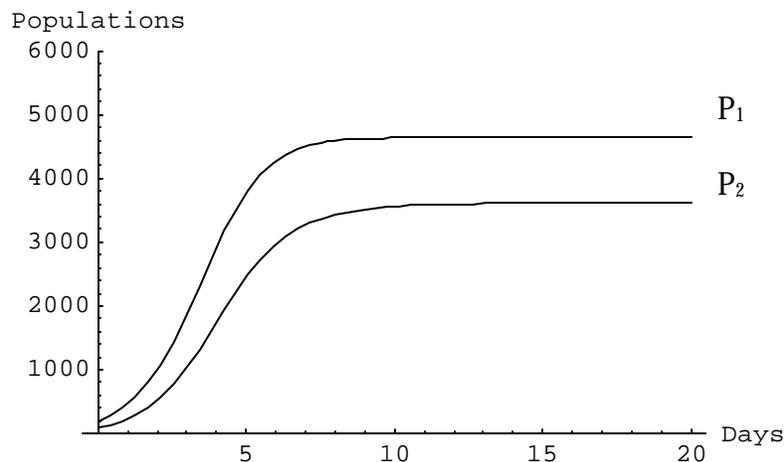


Figure 4.21 Growth curves of *Paramecium caudatum* and *Paramecium aurelia*.

The steady state or equilibrium condition can be solved by setting the differential equations to zero:

$$\begin{aligned}\frac{dP_1}{dt} &= r_1 P_1 \left(1 - \frac{P_1}{L_1}\right) - a_1 P_1 P_2 = 0 \\ \frac{dP_2}{dt} &= r_2 P_2 \left(1 - \frac{P_2}{L_2}\right) - a_2 P_1 P_2 = 0\end{aligned}\tag{4.16}$$

which leads to:

$$\begin{aligned}P_2 &= \frac{r_1}{a_1} \left(1 - \frac{P_1}{L_1}\right) \\ P_2 &= L_2 \left(1 - \frac{a_2 P_1}{r_2}\right)\end{aligned}\tag{4.17}$$

```
r1=0.9; L1=5000; r2=0.6; L2=2000;
a1=r1/L1 + r1/L1*3.3;
a2=r2/L2 - r2/L2*0.08;
P2eq1=r1/a1*(1-P1/L1);
P2eq2=L2(1-P1*a2/r2);
p5 = Plot[{P2eq1, P2eq2}, {P1,0,6000}, PlotRange -> {{0,6000},{0,3000}},
  AxesLabel -> {"P. paramecium", "P. aurelia"}];
```

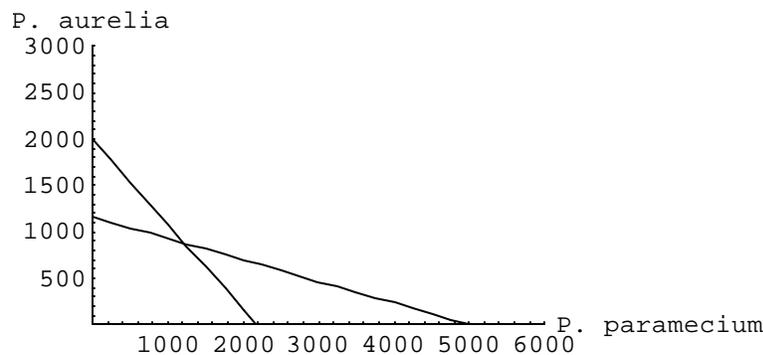


Figure 4.22. Isoclines of  $\frac{dP_1}{dt} = 0$  and  $\frac{dP_2}{dt} = 0$ .

Isocline Analysis

```

r1=0.9; L1=5000; r2=0.6; L2=2000;
a1=r1/L1 + r1/L1*1.3;
a2=r2/L2 - r2/L2*0.06;
population[x_]:=Module[ {solt, P1,P2,t,pop3,pop4},
  pop3=P1'[t] == r1*P1[t] (1-P1[t]/L1) - a1*P1[t]*P2[t];
  pop4=P2'[t] == r2*P2[t] (1-P2[t]/L2) - a2*P1[t]*P2[t];
  solt=NDSolve[{pop3, pop4, P1[0]==10x, P2[0]==5x}, {P1[t], P2[t]}, {t,0,200}];
  ParametricPlot[{P1[t],P2[t]} /. solt, {t,0,200}, Compiled->False,
  DisplayFunction->Identity ] ];
graphs=Table[population[z], {z,5,500,100}]
p6=Show[p5, graphs, PlotRange->{{0,5000},{0,3000}}, AxesLabel -> {"P. caudatum","P. aurelia"}, DisplayFunction->$DisplayFunction]

```

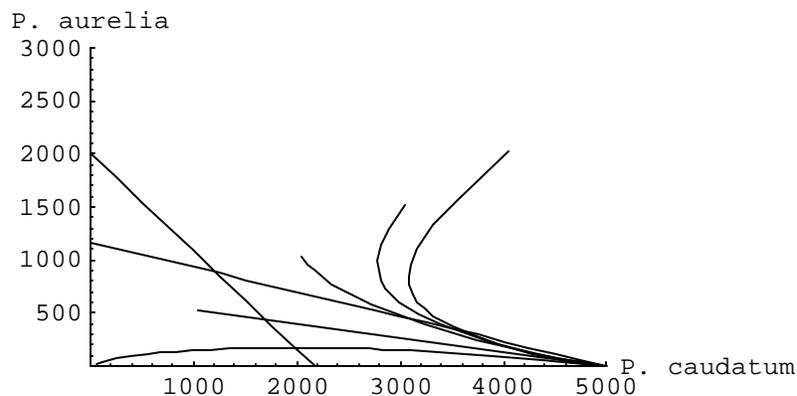


Figure 4.23. Illustration of the stable point.

By increasing the carrying capacity of P1 and changing the competitive advantage factors, the stable equilibrium point changes as shown below:

```

r1=0.9; L1=5500; r2=0.6; L2=2000;
a1=r1/L1 + r1/L1*3.3;
a2=r2/L2 - r2/L2*0.6;
population[x_]:=Module[ {solt, P1,P2,t,pop3,pop4},
  pop3=P1'[t] == r1*P1[t] (1-P1[t]/L1) - a1*P1[t]*P2[t];
  pop4=P2'[t] == r2*P2[t] (1-P2[t]/L2) - a2*P1[t]*P2[t];
  solt=NDSolve[{pop3, pop4, P1[0]==10x, P2[0]==5x}, {P1[t], P2[t]}, {t,0,200}];
  ParametricPlot[{P1[t],P2[t]} /. solt, {t,0,200}, Compiled->False,
  DisplayFunction->Identity ] ];
graphs=Table[population[z], {z,5,500,100}]
p7=Show[p5, graphs, PlotRange->{{0,5000},{0,3000}}, AxesLabel -> {"P. caudatum","P. aurelia"}, DisplayFunction->$DisplayFunction]

```

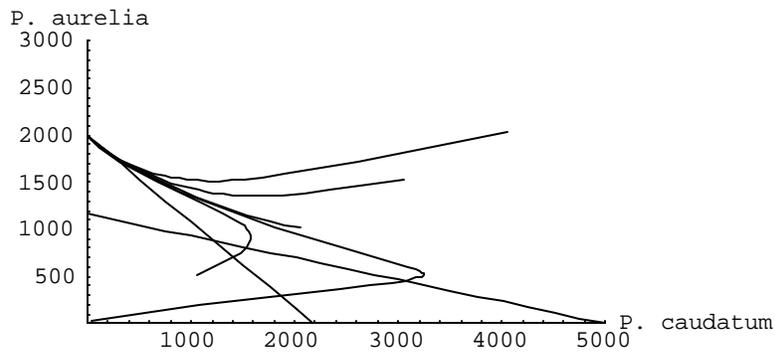


Figure 4.24. Illustration of the stable point.

Show[p6, p7]

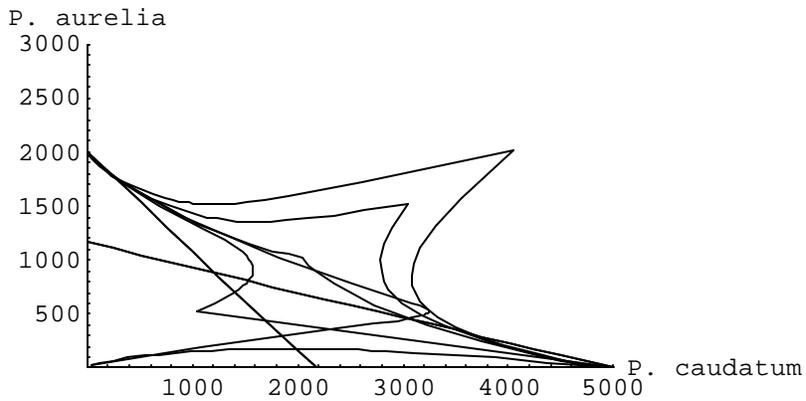


Figure 4.25. Illustration of two stable points.

## 4.7 The SIRS MODEL

We studied the SIRS model on epidemics from equation (4.12). The SIRS equations are reviewed as follows:

$$\frac{dS}{dt} = -bSI + gR \tag{4.18a}$$

$$\frac{dI}{dt} = bSI - nI \tag{4.18b}$$

$$\frac{dR}{dt} = nI - gR. \tag{4.18c}$$

Equation (4.18) has two steady states:

$$\bar{S}_1 = N, \quad \bar{I}_1 = 0, \quad \bar{R}_1 = 0 \quad (4.19)$$

$$\bar{S}_2 = \frac{\mathbf{n}}{\mathbf{b}}, \quad \bar{I}_2 = \mathbf{g} \frac{N - \bar{S}_2}{\mathbf{n} + \mathbf{g}}, \quad \bar{R}_2 = \frac{\mathbf{n}\bar{I}_2}{\mathbf{g}}. \quad (4.20)$$

In equation (4.19), the whole population is healthy and the disease is eradicated. In equation (4.20) the population consists of some constant proportions of each type provided  $(\bar{S}_2, \bar{I}_2, \bar{R}_2)$  are all positive quantities. For  $\bar{I}_2$  to be positive,  $N$  must be larger than  $\bar{S}_2$ . Since  $\bar{S}_2 = \mathbf{n}/\mathbf{b}$ , then for the disease to be established the total population  $N$  must exceed the level  $\mathbf{n}/\mathbf{b}$ , that is,  $\frac{N}{S} = \frac{N\mathbf{b}}{\mathbf{n}} > 1$ . That is, the population must be “large enough” for a disease to become endemic.

The ratio of parameters  $\beta/v$  has a rather meaningful interpretation. Since removal rate from the infective class is  $v$  (in units of 1/time), the average period of infectivity is  $1/v$ . Thus  $\beta/v$  is the fraction of the population that comes into contact with an infective individual during the period of infectiousness. The quantity  $R_0 = N\beta/v$  has been called the infectious contact number (Hethcote, 1976) or the intrinsic reproductive rate of the disease (May, 1983).  $R_0$  represents the average number of secondary infections caused by introducing a single infected individual into a host population of susceptibles.

In further analyzing the model we can take into account the particularly convenient fact that the total population

$$N = S + I + R \quad (4.21)$$

does not change. This means that one variable, say  $R$ , can be eliminated so that the model can be given in terms of two equations in two unknowns. Since the total population is constant, we eliminate equation (4.18c) and substitute  $R = N - S - I$  into equation (4.18a). The equations for  $S$  and  $I$  then become

$$\begin{aligned} \frac{dS}{dt} &= -\mathbf{b}SI + \mathbf{g}(N - S - I) \\ \frac{dI}{dt} &= \mathbf{b}SI - \mathbf{n}I \end{aligned} \quad (4.22)$$

The steady state solution is

$$\bar{S} = \frac{\mathbf{n}}{\mathbf{b}}, \quad \bar{I} = \mathbf{g} \frac{N - \bar{S}}{\mathbf{b}\bar{S} + \mathbf{g}} \quad (4.23)$$

## Exercises

1. From Example 4.1, estimate the vole population over 10 years if the initial conditions are: 10, 20, 40, 60, 120, 140, 160, and 200. Show results in one graph.
2. Estimate the population of *Paramecium caudatum* (Pascual and Kareiva, 1996) after 20 days if the growth rate  $r$  is  $0.9 \text{ d}^{-1}$ , the carrying capacity is 5500 *P. caudatum* cultures, and the initial population is 20.
3. *Gompertz law*. Another growth model is the Gompertz law which is often used to depict the growth of solid tumors (Edelstein-Keshet, 1988). Because cells in the interior of a tumor may not have ready access to nutrients and oxygen, it is assumed that the growth rate of the tumor declines as the cell mass grows. Edelstein-Keshet (1988) presented three equivalent versions of this growth rate as follows:

$$\frac{dP}{dt} = \lambda e^{-\alpha P} P \quad (4.24)$$

$$\frac{dP}{dt} = \mathbf{g}P, \quad \frac{d\mathbf{g}}{dt} = -\mathbf{a}\mathbf{g} \quad (4.25)$$

$$\frac{dP}{dt} = -\mathbf{k}P \ln P \quad (4.26)$$

If  $\lambda=0.06$ ,  $\alpha=0.02$ , and  $\kappa=0.06$ , show the growth curves of equations (4.24) through (4.26) over 50 weeks. Which equation demonstrates an exponential function?

4. The predator-prey model in Example 4.2 can be modified to allow the prey ( $P_1$ ) to move in and out of its habitat at a constant rate  $s$ , where  $s>0$  means migration out or harvesting by other agents;  $s<0$  means immigration into the habitat. This scenario leads to the following model:

$$\begin{aligned} \frac{dP}{dt} &= rP_1 \left(1 - \frac{P_1}{L}\right) - aP_1P_2 + s \\ \frac{dP}{dt} &= -cP_2 + bP_1P_2 \end{aligned}$$

Use the parameters already defined. Let  $s = 2 \text{ voles ha}^{-1}$ .

- a. Analyze the predator-prey system when  $c = 2.5$ . Discuss your results.
  - b. Analyze the predator-prey system when  $c = 0.532807$ . Discuss your results.
5. Do a predator-prey analysis of the voles as prey (R) and the weasels as predators (M) (Hanski et al., 1993). Use the following equations:

$$\frac{dP_1}{dt} = rP_1 \left( 1 - \frac{P_1}{L} \right) - \frac{a P_1 P_2}{P_1 + D}$$

$$\frac{dP_2}{dt} = vP_2 \left( 1 - \frac{qP_2}{P_1} \right)$$

where  $r = 5.4$  rodents  $\text{yr}^{-1}$ ;  $L=100$  rodents  $\text{ha}^{-1}$ ;  $a=600$  (consumption rate of weasels on voles);  $v=2.8$  weasels  $\text{km}^{-2}$ ;  $q=50, 100, 200$ ;  $P_1/q$ =carrying capacity of weasels;  $D=2.5, 5, 20$  (mortality of weasels). Let the initial populations be as follows:  $P_1[0] = 10$ ;  $P_2[0] = 10$ .

6. *Fish-fishermen dynamics.* (Edelstein-Keshet, 1988, p. 265). Consider a lake with some fish attractive to fishermen. We shall model the fish-fishermen interaction.

*Fish Assumptions:*

- (i) Fish grow logistically in the absence of fishing.
- (ii) The presence of fishermen depresses fish growth at a rate jointly proportional to the fish and fishermen populations.

*Fishermen Assumptions:*

- (i) Fishermen are attracted to the lake at a rate directly proportional to the amount of fish in the lake.
  - (ii) Fishermen are discouraged from the lake at a rate directly proportional to the number of fishermen already there.
- (a) Formulate, analyze, and interpret a mathematical model for this situation.
  - (b) Suppose the department of fish and game decides to stock the lake with fish at a constant rate. Formulate, analyze, and interpret a mathematical model for the situation with stocking included. What effect does stocking have on the fishery?

7. *Whale-krill dynamics.* (Edelstein-Keshet, 1988, p. 265). Beddington and May (1982) have proposed the following model to study the interactions between baleen whales and their main food source, *krill* (a small shrimp-like animal), in the southern ocean:

$$\text{krill: } \dot{x} = rx \frac{1-x}{K} - axy$$

$$\text{whales: } \dot{y} = sy \frac{1-y}{bx}$$

Here the whale carrying capacity is not constant but is a function of the krill population:

$$K_{whales} = bx$$

Analyze this model by determining the steady states and their stability. Include a phase-plane diagram.

8. With reference to Example 4.3, run the epidemic model using the following values:
  - i.  $a=0.4$ ,  $b=.1$ ; interpret your results.
  - ii.  $a=0.6$ ,  $b=0.3$ ; interpret your results.
9. Consider the problem where  $P_1$  and  $P_2$  are food sources to  $P_3$ , such as pheasants ( $P_1$ ) and turkeys ( $P_2$ ) as food sources to foxes ( $P_3$ ). On the other hand,  $P_1$  and  $P_2$  compete for the same food source. Use the following model:

$$\begin{aligned}\frac{dP_1}{dt} &= r_1 P_1 \left(1 - \frac{P_1}{L_1}\right) - a_1 P_1 P_2 - h P_1 P_3 \\ \frac{dP_2}{dt} &= r_2 P_2 \left(1 - \frac{P_2}{L_2}\right) - a_2 P_1 P_2 - j P_2 P_3 \\ \frac{dP_3}{dt} &= -c P_3 + d P_1 P_3 + j P_2 P_3\end{aligned}$$

where  $r_1$ ,  $r_2$ ,  $L_1$ ,  $L_2$ ,  $a_1$ , and  $a_2$  are parameters as defined in the Competition model;  $c$  is the per-capita death rate of the predator;  $h$  is the per-capita rate at which  $P_3$  captures  $P_1$ ;  $j$  is the per-capita rate at which  $P_3$  captures  $P_2$ ;  $d$  is the per-capita growth of  $P_3$  resulting from the capture of  $P_1$ ;  $g$  is the per-capita growth of  $P_3$  resulting from the capture of  $P_2$ . Values of  $j$  that have been tried yield the following observations:  $j=0.001009$  shows that  $P_1$  eventually survives and  $P_2$  eventually dies;  $j=0.001$  shows that  $P_1$  dies and  $P_2$  survives;  $j=0.001005$  shows that  $P_1$  eventually survives and  $P_2$  survives;  $j=0.001008$  shows that  $P_1$  eventually survives and  $P_2$  survives.

10. Consider a predator-prey system where the predator satiates. Let the pheasants be the prey and the foxes be the predator and their respective populations be represented as  $P_1$  and  $P_2$ . Use the following model:

$$\begin{aligned}\frac{dP_1}{dt} &= r_1 P_1 \left(1 - \frac{P_1}{L_1}\right) - \frac{b P_1 P_2}{a + P_1} \\ \frac{dP_2}{dt} &= r_2 P_2 \left(1 - \frac{P_2}{c P_2}\right)\end{aligned}$$

where  $cP_1$  is the carrying capacity for  $P_2$  (which is a function of the available food source,  $P_1$ ). Use the following values:  $r_1=0.00034$ ;  $L_1=2886$ ;  $r_2=0.0004$ ;  $a=0.001$ ;  $c=1.0$ ;  $b=0.0001$ .

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## Conservation of Mass in Natural Resource Systems

### PRINCIPLE

At steady state, input rate is equal to output rate.

### TOPICS

- One-compartment system
- Two-compartment system
- Three-compartment system
- Multiple-compartment system

Conservation of mass is a basic principle in biological and environmental systems modeling. Also known as the input-output modeling, this technique shows that the rate of change of the concentration of a substance in a closed system in a given interval of time  $\Delta t$  is due to the rate at which the substance enters the system minus the rate at which it leaves the system in the interval of time. That is,

$$\text{rate of change of the concentration} = (\text{input rate} - \text{output rate})\Delta t$$

A closed system means that there are no other sources or sinks of the substance. The biosystem is modeled as a compartment. Two approaches are used to solve the compartment problem: (1) continuous time solution of a differential equation and (2) steady state method. The exercises are simplified illustrations of real-world biologically-based systems and designed to develop in the students the ability to translate “word problems” into quantitative statements, evaluate the validity of assumptions, apply appropriate analytical tools to solve the problem, and explore the consequences of changing initial conditions.

### 5.1 One-Compartment System

The idealized one-compartment dilution system consists of a single continuously mixed chamber through which a fluid is flowing at a constant rate. One can visualize this as a bucket with a hole in the bottom from which fluid is flowing, and a garden hose is keeping the bucket filled to a fixed level.

#### Example 5.1: Lake pollution

Nutrients, such as nitrogen (N) and phosphorus (P), which enter surface waters in concentrated levels, can stimulate rapid aquatic plant growth. Such rapid growth chokes surface waters with weeds and algae, deplete oxygen levels, and kill certain species of aquatic organisms, resulting in changes in the ecological composition of water habitat (Michigan United Conservation Clubs, 1993).

Phosphorus is a major chemical ingredient in fertilizers as it is required in crop growth. As shown in the phosphorus biochemical cycle (Fig. 2.3), a load of phosphorus compounds from sediment runoff or waste discharge could end up in lakes and rivers, causing major water pollution.

Consider the problem of a phosphorus-rich sediment being dumped into a lake at a rate of  $0.16 \text{ tons P d}^{-1}$ , as shown in schematic in Fig. 5.1 (Harte, 1985). Assume that the P compound is stable, highly soluble, and uniformly mixed in the lake. The lake volume is given to be  $4 \times 10^7 \text{ m}^3$  and the average water flow-through rate is  $8 \times 10^4 \text{ m}^3 \text{ d}^{-1}$ . What is the concentration of the P in the lake over time, ignoring evaporation from the lake surface? What is the equilibrium concentration of the pollutant P?

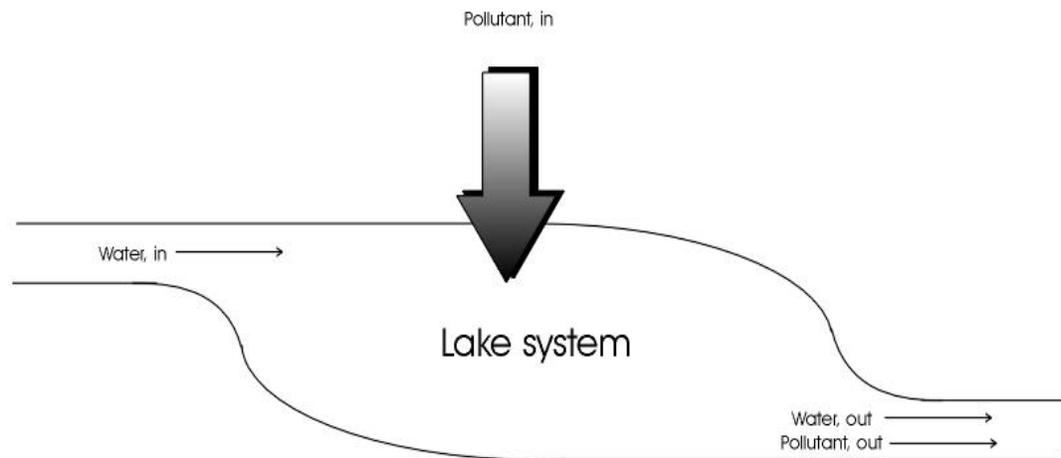


Figure 5.1. Schematic diagram of a lake with pollution input.

To solve the problem, proceed as follows:

**Step 1.** *What is being asked?* Find the pollutant concentration over time,  $C(t)$ . Concentration is the ratio of mass to volume.

**Step 2.** *What information are given?*

$$\text{volume of water in the lake (V)} = 4 \times 10^7 \text{ m}^3$$

$$\text{water flow rate (F}_w) = 8 \times 10^4 \text{ m}^3 \text{ d}^{-1}$$

$$\text{pollutant flow rate (F}_p) = 0.16 \text{ t d}^{-1}$$

The net change in pollutant concentration in the interval of time  $\Delta t$  is given as:

$$\Delta C = (\text{input rate} - \text{output rate}) \Delta t$$

The input rate of pollutant concentration in the interval of time  $\Delta t$  is  $\frac{F_p}{V} \Delta t$

The output rate of pollutant concentration in the interval of time  $\Delta t$  is  $-\frac{F_w}{V} C(t) \Delta t$

The net change in pollutant concentration in the interval of time  $\Delta t$  is given as

$$\Delta C = \left( \frac{F_p}{V} - \frac{F_w}{V} C \right) \Delta t$$

Dividing both sides of the equation by  $\Delta t$ , we have:

$$\frac{\Delta C}{\Delta t} = \left( \frac{F_p}{V} - \frac{F_w}{V} C \right)$$

Taking the limit as  $\Delta t \rightarrow 0$ , we get the derivative of  $C$  with respect to  $t$ :

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta C}{\Delta t} = \frac{dC}{dt} = \frac{F_p}{V} - \frac{F_w}{V} C$$

Using the DSolve function in Mathematica for  $C(0) = 0$  (zero initial condition), the solution is:

```
DSolve[{C'[t] == Fp/V - Fw/V*C[t], C[0]==0}, C[t], t]
```

$$\left\{ \left\{ C[t] \rightarrow \frac{-F_p + E \frac{(F_w t)/V}{F_w}}{(F_w t)/V} \right\} \right\}$$

In simplified terms,

$$C(t) = \frac{F_p}{F_w} \left( 1 - e^{-\frac{F_w}{V} t} \right) \quad (5.1)$$

Equation (5.1) is the mathematical model that will allow for the calculation of pollutant concentration of a lake at any time, as a function of water flow rate, pollutant flow rate, and lake volume.

How is the problem solved using the steady state approach? At steady state, there are two assumptions: (1) after a long enough time (represented as time infinity), the system goes into a state of equilibrium and (2) at equilibrium, input is

equal to output since the rate of change is zero. In cases where the differential equation is available, such as the problem above, the derivative is set equal zero and the concentration is solved as a function of infinite time, that is,

$$\frac{F_p}{V} = \frac{F_w}{V} C$$

$$C = \frac{F_p}{F_w} = \frac{0.16t/d}{8 \times 10^4 m^3/d} = 2 \times 10^{-6} t/m^3$$

When the solution is available, such as also the case above, the variable time  $t$  is set equal to infinity and the concentration is solved as a function of infinite time, that is,

$$C(\infty) = \frac{F_p}{F_w} \left( 1 - e^{-\frac{F_w}{V} \infty} \right)$$

$$C(\infty) = \frac{F_p}{F_w} (1 - 0)$$

$$C(\infty) = \frac{F_p}{F_w}$$

If neither the derivative nor the solution form is available, begin by solving for the residence time. For this problem, start by finding the residence time of the pollutant  $P$  and then calculate the steady-state concentration of  $P$  using the information of the calculated residence time.

Let  $T_w$  and  $T_p$  be the residence time of water and pollutant, respectively and  $M$  be the mass of the pollutant. Then, the residence time of water is

$$T_w = V/F_w = 500 \text{ days.}$$

We assume that the pollutant is well mixed in the water, and so  $T_w = T_p$ . Then

$$M = F_p T_p = 80 \text{ t.}$$

and

$$C = M/V = 2 \times 10^{-6} \text{ t m}^{-3}.$$

Note that  $1 \text{ m}^3$  of water weighs 1 ton, and so,  $C = 2 \times 10^{-6} = 2$  parts per million (ppm) by weight.

Suppose evaporation cannot be ignored, so that total rate at which water exits the lake has two components: evaporation (one third) and stream outflow (two thirds), as illustrated in Fig. 5.2. The total rate at which water exits the lake is unchanged. Assume that the evaporating water is free of pollutant. What is the new steady-state concentration of the pollutant? One reason why algae blooms occur in summer!

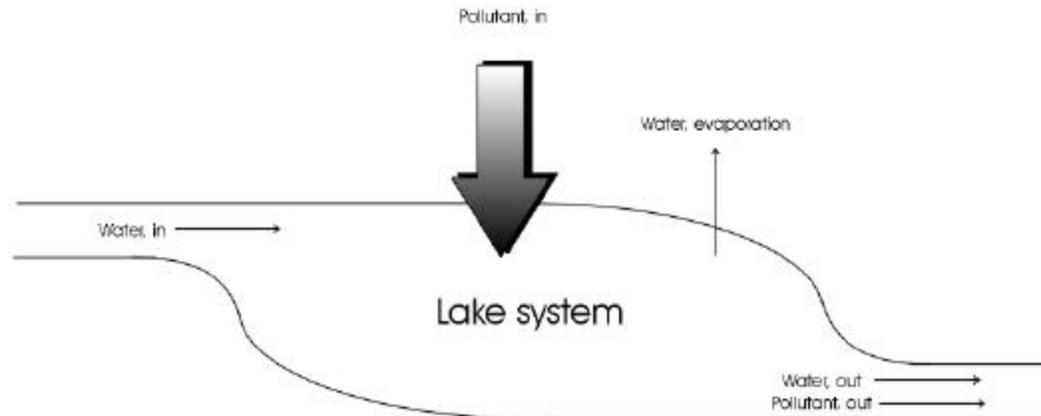


Figure 5.2. Schematic representation of a polluted lake with evaporation.

## 5.2 Two-Compartment System

It is common for two compartments to be in series. This phenomenon is true of lakes forming along the same river flow.

### Example 5.2. A Network of Polluted Lakes

Consider the problem of two lakes which are connected by the same river, as shown in Fig. 5.3. Lake 1 is upstream of lake 2. Water flow-through (input and output) for both lakes is  $F_W$ . Water volume of lake 1 is given as  $V_1$  and water volume of lake 2 is given as  $V_2$ . Assume that evaporation from the lake surface can be ignored. A P-rich sediment is added into lake 1. What is the pollutant P concentration ( $C_1$ ) in lake 1? What is the pollutant P concentration ( $C_2$ ) in lake 2? Suppose lake 1 is that lake described in Exercise 5.1 but the pollutant is dumped only once at the beginning (time zero). Find  $C_1$  and  $C_2$  over time. Show graphically the relationship of  $C_1(t)/C_1(0)$ ,  $C_2(t)/C_1(0)$  for  $V_1 = 10V_2$ ,  $C_2(t)/C_1(0)$  for  $V_1 = V_2$ , and  $C_2(t)/C_1(0)$  for  $V_2 = 10 \cdot V_1$ .

To solve the problem, proceed as follows:

1. *Solve for  $C_1(t)$ :*

- $\Delta C_1 = (\text{input rate} - \text{output rate})\Delta t$ ; where input rate = 0

$$\Delta C_1 = \frac{F_{w1}}{V_1} C_1 \Delta t$$

- The solution is:

$$\begin{aligned} \frac{\Delta C_1}{\Delta t} &= -\frac{F_{w1}}{V_1} C_1 \\ \lim_{\Delta t \rightarrow 0} \frac{\Delta C_1}{\Delta t} &= \frac{dC_1}{dt} = -\frac{F_{w1}}{V_1} C_1 \\ \int_0^t C_1 &= -\frac{F_{w1}}{V_1} \int_0^t dt \\ \Rightarrow \ln \frac{C_1(t)}{C_1(0)} &= -\frac{F_{w1}}{V_1} t \\ \Rightarrow C_1(t) &= C_1(0) e^{-\frac{F_{w1}}{V_1} t} \\ \Rightarrow C(t) &= \frac{P_0}{V_1} e^{-\frac{F_{w1}}{V_1} t} \end{aligned}$$

2. Solve for  $C_2(t)$ :

- Amount of pollutant entering lake 2 in the interval of time,  $\Delta t$ :

$$P_{2i} = F_{w1} C_1 \Delta t$$

- Amount of pollutant leaving lake 2:

$$P_{2o} = -F_{w2} C_2 \Delta t$$

- Change in amount of pollutant in the interval of time  $\Delta t$ :

$$\Delta P = [F_{w1} C_1 - F_{w2} C_2] \Delta t$$

- Change in concentration:

$$\begin{aligned} \Delta C_2 &= \left[ \frac{F_{w1}}{V_2} C_1 - \frac{F_{w2}}{V_2} C_2 \right] \Delta t \\ \lim_{\Delta t \rightarrow 0} \frac{\Delta C_2}{\Delta t} &= \frac{dC_2}{dt} = \frac{F_{w1}}{V_2} C_1 - \frac{F_{w2}}{V_2} C_2 \end{aligned}$$

- But:

$$F_{w1} = F_w = F_{w2}$$

- So:

$$\frac{dC_2}{dt} = \frac{F_w}{V_2} C_1 - \frac{F_w}{V_2} C_2 = \frac{F_w}{V_2} [C_1 - C_2]$$

- Substitute  $C_1(t)$ :

$$\frac{dC_2}{dt} = \frac{F_w}{V_2} \left[ C_1(0) e^{-\frac{F_w}{V_1} t} - C_2(t) \right]$$

- Solving by Mathematica, we have:

$$C_2(t) = \frac{C_1(0)V_1}{(V_1 - V_2)} e^{-\frac{F_w}{V_1} t} + \frac{C_1(0)V_1}{(-V_1 + V_2)} e^{-\frac{F_w}{V_2} t}$$

- Note that:  $\frac{a}{(-a+b)} = -\frac{a}{(a-b)}$

$$\begin{aligned} C_2(t) &= \frac{C_1(0)V_1}{(V_1 - V_2)} e^{-\frac{F_w}{V_1} t} - \frac{C_1(0)V_1}{(V_1 - V_2)} e^{-\frac{F_w}{V_2} t} \\ \Rightarrow C_2(t) &= \frac{C_1(0)V_1}{(V_1 - V_2)} \left[ e^{-\frac{F_w}{V_1} t} - e^{-\frac{F_w}{V_2} t} \right] \end{aligned}$$

$$(a) \quad \frac{C_1(t)}{C_1(0)} = e^{-\frac{F_w}{V_1} t}$$

$$(b) \quad \frac{C_2(t)}{C_1(0)} = \frac{V_1}{(V_1 - V_2)} \left[ e^{-\frac{F_w}{V_1} t} - e^{-\frac{F_w}{V_2} t} \right]$$

- Suppose a third lake is downstream of lake 2. Write the differential equation that describes the pollutant concentration in lake 3.

*Solution:*

$$\frac{dC_3}{dt} = \frac{F_w}{V_3} [C_2 - C_3]$$

where  $C_2(t)$  is as derived earlier.

$$\Rightarrow \frac{dC_3}{dt} = \frac{F_w}{V_3} \left[ \frac{C_1(0)V_1}{(V_1 - V_2)} \left( e^{-\frac{F_w t}{V_1}} - e^{-\frac{F_w t}{V_2}} \right) - C_3 \right]$$

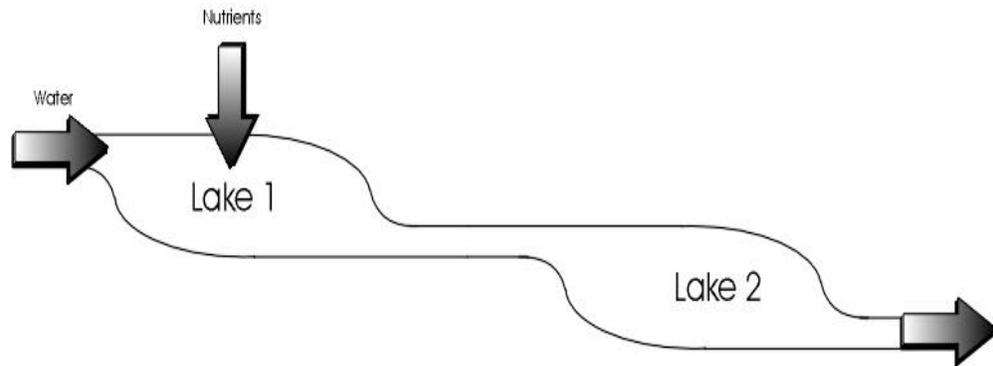


Figure 5.3. Schematic diagram of two lakes in sequence.

*✎* Example 5.3. Connected Lakes with a Tributary

Consider two lakes along the same river, as shown in Fig. 5.4. A is upstream of B. Water flows into A at a rate  $S_A$ ; it evaporates from A at a rate  $E_A$  and from B at a rate  $E_B$  (Harte, 1985, pp. 36-37). A tributary flows into the river between A and B at a rate  $S_B$ . Evaporation from the streams can be ignored. Phosphates (as pollutant) flow into lake A at a rate  $P$ . There are no other sources of the pollutant, it is well mixed in both lakes, and it does not codistill. The lakes have water volume  $V_A$  and  $V_B$ , respectively, and are in hydrological steady states.

- (a) What is the stream flow rate out of lake A? Into lake B?
- (b) What is the residence time of water in lake A? In lake B?

At steady state, what concentration of pollutant will be found in each lake?

**Solution**

1.  $F_{A,out} = S_A - E_A$

$F_{B,in} = F_{A,out} + S_B = S_A - E_A + S_B$  ← flows into lake B include the flows into and out of lake A.

$$2. \quad T_A = \frac{V_A}{S_A - E_A} = \frac{V_A}{F_{A,out}}$$

$$T_B = \frac{V_B}{F_{B,in} - E_B} = \frac{V_B}{S_A - E_A + S_B - E_B}$$

3.

$$(a) \quad \Delta C_1 = \left( \frac{F_P}{V_A} - \frac{F_{A,out}}{V_A} C_1 \right) \Delta t \Leftrightarrow \Delta C_1 = (\text{inflow rate} - \text{outflow rate}) \Delta t$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta C_1}{\Delta t} = \frac{F_P}{V_A} - \frac{F_{A,out}}{V_A} C_1 \Rightarrow \frac{dC_1}{dt} = \frac{F_P}{V_A} - \frac{F_{A,out}}{V_A} C_1$$

At steady state:

$$\frac{dC_1}{dt} = 0 \Rightarrow C_1 = \frac{F_P}{V_A} \frac{V_A}{F_{A,out}} = \frac{F_P}{F_{A,out}}$$

$$C_1 = \frac{F_P}{S_A - E_A}$$

$$(b) \quad \Delta C_2 = (\text{inflow rate} - \text{outflow rate}) \Delta t = \left( \frac{F_{A,out}}{V_B} C_1 - \frac{F_{B,out}}{V_B} C_2 \right) \Delta t$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta C_2}{\Delta t} = \frac{dC_2}{dt} = \frac{F_{A,out}}{V_B} C_1 - \frac{F_{B,out}}{V_B} C_2$$

At steady-state:

$$\frac{dC_2}{dt} = 0 \Rightarrow C_2 = \frac{F_{A,out}}{V_B} C_1 \frac{V_B}{F_{B,out}} = \frac{F_{A,out}}{F_{B,out}} \frac{F_P}{F_{A,out}} = \frac{F_P}{F_{B,out}}$$

$$C_2 = \frac{F_P}{F_{B,in} - E_B} = \frac{F_P}{S_A - E_A + S_B - E_B}$$

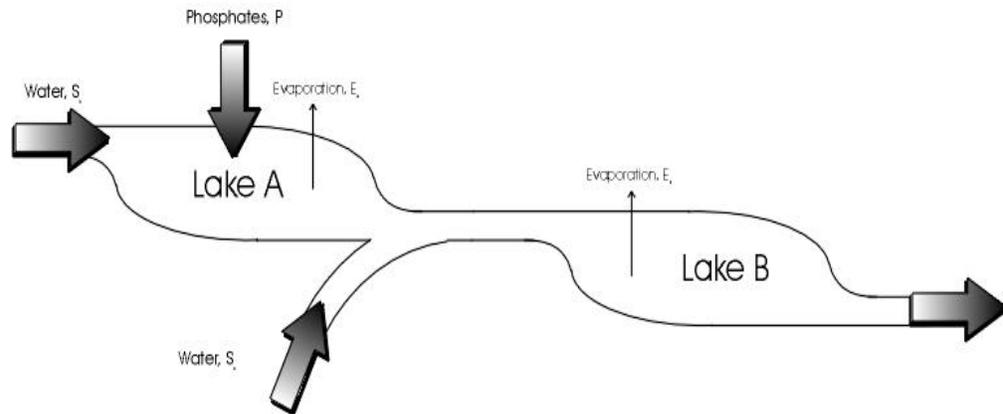


Figure 5.4. Connected lakes with a tributary.

### 5.3 Three-Compartment System

#### ✍ Example 5.4. Phosphorus Cycling in a Lake

In 1987, phosphorus loading for the Saginaw Bay was estimated at 665 metric tons, 50 percent higher than the target loading of 440 metric tons per year by the Great Lakes Water Quality Agreement (Michigan United Conservation Clubs, 1993). Consider the problem of inorganic P compounds in a lake being taken up by phytoplanktons to form P-rich biomass. These planktons die and the P compounds are converted into dead organic matter. The organic matter is mineralized into inorganic form. Let  $X_1$  represent the amount of P in living biomass,  $X_2$  represent the amount of P in inorganic form, and  $X_3$  represent the amount of P in dead organic matter (Harte, 1985, pp. 45-49). Each  $X_i$  is in units of micromoles of P per liter of lake water. Let  $F_{ij}$  be the flow of P from stock  $i$  to stock  $j$ . Fig. 5.5 shows a schematic diagram of phosphorus (P) cycling in a lake, where:

$$X_1(0) = 0.2 \text{ } \mu\text{moles P li}^{-1} \text{ (amount of P in living biomass)}$$

$$X_2(0) = 0.1 \text{ } \mu\text{moles P li}^{-1} \text{ (amount of P in inorganic form)}$$

$$X_3(0) = 1.0 \text{ } \mu\text{moles P li}^{-1} \text{ (amount of P in dead organic material)}$$

At time  $t=0$ , the system is perturbed by a sudden addition of  $0.02 \text{ } \mu\text{moles P li}^{-1}$  (such as in the form of animal manure). What is the behavior of the perturbed system over time? Derive the differential equations and estimate the amount of P in each compartment using Mathematica. Using the steady-state approach and given that the residence time of P in living biomass is 4 days, how much P will be in each compartment at steady state?

**Solution:**

- Derive the differential equations:

$$\Delta X = (\text{inflow rate} - \text{outflow rate}) \Delta t$$

$$\frac{\Delta X}{\Delta t} = \text{inflow rate} - \text{outflow rate}$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta X}{\Delta t} = \frac{dX}{dt} = \text{inflow rate} - \text{outflow rate}$$

$$(a) \quad \frac{dX_1}{dt} = \beta X_2 X_1 - \gamma X_1$$

$$(b) \quad \frac{dX_2}{dt} = \alpha X_3 - \beta X_2 X_1$$

$$(c) \quad \frac{dX_3}{dt} = \gamma X_1 - \alpha X_3$$

- What are the values of  $\alpha$ ,  $\beta$ , and  $\gamma$ ?
- What is the steady-state condition before the perturbation?

$$\beta X_2 X_1 = \gamma X_1 \quad \rightarrow \quad X_2 = \frac{\gamma X_1}{\beta X_1} = \frac{\gamma}{\beta}$$

$$\alpha X_3 = \beta X_2 X_1 \quad \rightarrow \quad X_3 = \frac{\beta X_2 X_1}{\alpha}$$

$$\gamma X_1 = \alpha X_3 \quad \rightarrow \quad X_1 = \frac{\alpha X_3}{\gamma}$$

$$T_{X_1} = \frac{\text{Conc.}_{x_1}}{F_{out, x_1}} = \frac{X_1}{\gamma X_1} = 4 \text{ days}$$

$$\Rightarrow \gamma = 0.25/\text{day}$$

$$\Rightarrow \beta = \frac{\gamma}{X_2} = \frac{0.25}{0.1} = \frac{\frac{2.5}{\text{day}}}{\frac{\text{mole}}{\text{li}}} = 2.5 \frac{\text{mole}}{\text{day li}}$$

$$\Rightarrow \alpha = \frac{\gamma X_1}{X_3} = \frac{(0.25)(0.2)}{1} = 0.05/\text{day}$$

When 0.02  $\mu\text{M}$ /li was added, the system was perturbed and then settled to an equilibrium. At equilibrium, how is the 0.02 distributed among the 3 components?

- By conservation of mass, total P before time 0 in the system is:

$$X_1 + X_2 + X_3 = 0.2 + 0.1 + 1.0 = 1.30$$

- At time zero:  $X_1 + X_2 + X_3 + 0.02 = 1.32$ .
- Then we define  $X'_1, X'_2, X'_3$  as the new steady-state.
- After perturbation, the new steady-state conditions are:

$$\mathbf{b}X'_2 X'_1 = \mathbf{g}X'_1$$

$$\mathbf{a}X'_3 = \mathbf{b}X'_2 X'_1$$

$$\mathbf{g}X'_1 = \mathbf{a}X'_3 \rightarrow X'_1 = \frac{\mathbf{a}}{\mathbf{g}} X'_3 = \frac{0.05}{0.25} X'_3 = 0.2 X'_3$$

$$\Rightarrow X'_2 = \frac{0.25 X'_1}{2.5 X'_1} = 0.1 \rightarrow X'_2 = 0.1$$

$$X'_3 = \frac{\mathbf{b}X'_2 X'_1}{\mathbf{a}} = \frac{2.5(0.1)X'_1}{0.05} = 5 X'_1$$

- By conservation of matter, since the system is closed:

$$X'_1 + X'_2 + X'_3 = 0.02 + X_1 + X_2 + X_3$$

$$\Rightarrow 0.02 + 0.2 + 0.1 + 1.0 = 1.32$$

$$\Rightarrow X'_1 + 0.1 + 5 X'_1 = 1.32$$

$$6 X'_1 = 1.32 - 0.1 = 1.22$$

$$X'_1 = \frac{1.22}{6} = 0.20333 \frac{\mu\text{molesP}}{\text{li}}$$

$$\Rightarrow X'_3 = 5 X'_1 = 5(0.20333) = 1.017 \frac{\mu\text{molesP}}{\text{li}}$$

$$\therefore X'_1 = 0.20333, X'_2 = 0.1, X'_3 = 1.017 \frac{\mu\text{molesP}}{\text{li}}$$

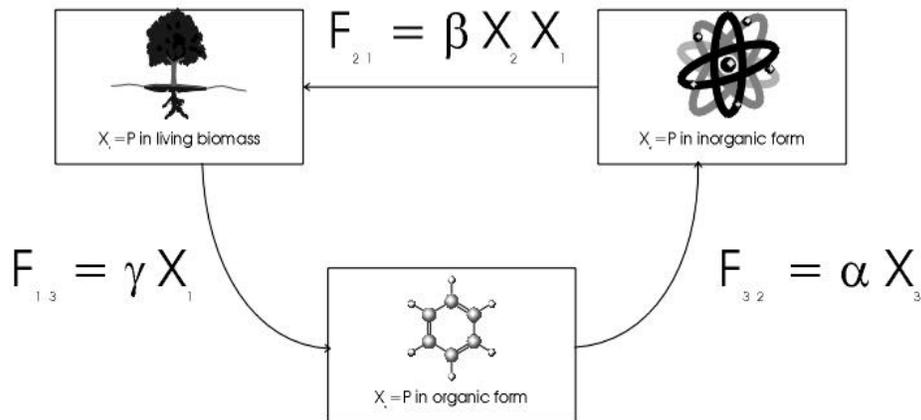


Figure 5.5. A schematic diagram of a phosphorus cycle.

Example 5.5. Phosphorus Cycling in a Lake, Revisited

Phosphorus cycling in the lake will be affected by the amount of phosphates dumped as pollutant or if acidic water (as acid rain) is added into the lake system. The conversion of P from organic to inorganic form is carried out by an enzyme called phosphatase (Harte, 1985, p.45). With reference to Example 5.4, suppose that at the initial  $t = 0$ , instead of a pollutant, acid rain is added into the lake which leads to the inhibition of phosphatase enzyme resulting in a 10% reduction of  $\alpha$ . When a new steady state is reached, how much P will be in each compartment?

Example 5.6. Circular reaction

The compounds A and  $A_1$  combine according to kinetic principles to produce B at a rate equal to the product of the concentrations of A and  $A_1$  and at a rate constant  $k_1$ . Then B reacts with  $B_1$  to form C at a rate constant  $k_2$ , and C reacts with  $C_1$  to replenish A at a rate constant  $k_3$ . If we let  $K_1 = k_1 \cdot A_1$ ,  $K_2 = k_2 \cdot B_1$ , and  $K_3 = k_3 \cdot C_1$ , the metabolic cycling can be represented graphically as given below.

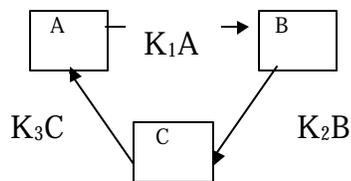


Figure 5.6. Three-component chemical cycle.

Assume that there is conservation of mass in the system, so that  $Q = A + B + C$

Let  $Q = 5.0 \text{ moles li}^{-1}$ ;  $K_1 = 0.4 \text{ s}^{-1}$  (rate constant);  $K_2 = 0.2 \text{ s}^{-1}$ ;  $K_3 = 2.0 \text{ s}^{-1}$ .

- Derive the differential equation for each compartment.
- Calculate the concentrations of A, B, and C compounds at steady state.

**Solution:**

- Derive the differential equations for each compartment.

$$\Delta A = (K_3 C - K_1 A) \Delta t \quad \rightarrow \quad \frac{\Delta A}{\Delta t} = \text{inflow rate} - \text{outflow rate}$$

$$\Delta B = (K_1 A - K_2 B) \Delta t$$

$$\Delta C = (K_2 B - K_3 C) \Delta t$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = K_3 C - K_1 A \rightarrow \frac{dA}{dt} = K_3 C - K_1 A$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta B}{\Delta t} = \frac{dB}{dt} = K_1 A - K_2 B \rightarrow \frac{dB}{dt} = K_1 A - K_2 B$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta C}{\Delta t} = \frac{dC}{dt} = K_2 B - K_3 C \rightarrow \frac{dC}{dt} = K_2 B - K_3 C$$

- At steady state,  $\frac{dA}{dt} = 0, \frac{dB}{dt} = 0, \frac{dC}{dt} = 0$

$$\Rightarrow K_3 C = K_1 A \rightarrow A = \frac{K_3 C}{K_1}$$

$$K_1 A = K_2 B \rightarrow B = \frac{K_1 A}{K_2}$$

$$K_2 B = K_3 C \rightarrow C = \frac{K_2 B}{K_3} = \frac{K_2}{K_3} \left( \frac{K_1 A}{K_2} \right) = \frac{K_1 A}{K_3}$$

By the conservation of mass principle, we have:

$$Q = A + B + C$$

$$Q = A + \frac{K_1 A}{K_2} + \frac{K_1 A}{K_3} = A \left[ 1 + \frac{K_1}{K_2} + \frac{K_1}{K_3} \right]$$

$$\Rightarrow A = \frac{Q}{1 + \frac{K_1}{K_2} + \frac{K_1}{K_3}} = \frac{5 \text{ moles / li}}{1 + \frac{0.4}{0.2} + \frac{0.4}{2}} = \frac{5}{3.2} = 1.5625 \text{ moles / li}$$

$$B = \frac{K_1 A}{K_2} = \frac{0.4}{0.2} * 1.5625 \text{ moles / li} = 3.125 \text{ moles / li}$$

$$C = \frac{K_1 A}{K_3} = \frac{0.4}{2} * 1.5625 \text{ moles / li} = 0.3125 \text{ moles / li}$$

$$\therefore A = 1.6 \text{ moles / li} \quad B = 3.1 \text{ moles / li} \quad C = 0.3 \text{ moles / li}$$

## 54 Multiple-component system

### **Example 5.7. The Hydrologic Cycle, Deforestation, and Drought**

Water is the source of all life on earth. The distribution of water, however, is quite varied; many locations have plenty of it while others have very little. Water exists on earth as a solid (ice), liquid or gas (water vapor). Oceans, rivers, clouds, and rain, all of which contain water, are in a frequent state of change (surface water evaporates, cloud water precipitates, rainfall infiltrates the ground, etc.). However, the total amount of the earth's water does not change. This example is about studying and modeling the effect of cutting trees from the earth's surface. Before we proceed, here are a few definitions for review. Evaporation is the transformation of water from a liquid into a gas, a process which humidifies the atmosphere. Condensation is the transformation of water from a gas into a liquid, and the processes that lead to condensation. Transport is the movement of water through the atmosphere. Precipitation is the transfer of water from the atmosphere to land. Rain, snow, hail, sleet, and freezing rain are discussed. Groundwater is the water located below ground and how it returns to the surface. Transpiration is the transfer of water to the atmosphere by plants and vegetation. Evapotranspiration is the combination of evaporation and transpiration processes. Runoff is the transport water from land to the oceans. Too much rainfall can cause excess runoff, or flooding. Deforestation is one major problem we are facing today. It is a process where trees and other vegetation are cleared from the land. This process results in desertification (the creation of deserts). If evapotranspiration from the Earth were to diminish by 20 percent uniformly over the land area due to deforestation, what changes would occur in the globally averaged precipitation on the land surface and in the globally averaged runoff from the land to the sea (Harte, 1985, pp. 50-54)? The following data are provided:

$$P_L \text{ (rate of precipitation on the land)} = 108 \times 10^3 \text{ km}^3 \text{ yr}^{-1}$$

$$P_S \text{ (rate of precipitation on the sea)} = 410 \times 10^3 \text{ km}^3 \text{ yr}^{-1}$$

$$R \text{ (rate of runoff from the land to the sea)} = 46 \times 10^3 \text{ km}^3 \text{ yr}^{-1}$$

$$E_{LL} \text{ (rate of evapotranspiration from the land that falls as precipitation on the land)} = 46.5 \times 10^3 \text{ km}^3 \text{ yr}^{-1}$$

$$E_{LS} \text{ (rate of evapotranspiration from the land that falls as precipitation on the sea)} = 15.5 \times 10^3 \text{ km}^3 \text{ yr}^{-1}$$

$$E_{SS} \text{ (rate of evaporation from the sea that falls as precipitation on the sea)} = 394.5 \times 10^3 \text{ km}^3 \text{ yr}^{-1}$$

$$E_{SL} \text{ (rate of evaporation from the sea that falls as precipitation on the land)} = 61.5 \times 10^3 \text{ km}^3 \text{ yr}^{-1}$$

Figure 5.7 shows a schematic representation of the system.

**Solution:**

Derive the differential equations:

$$\Delta P_L = (\text{input rate} - \text{output rate}) \Delta t$$

$$\Delta P_L = [P_L - (E_{LS} + E_{LL} + R)]\Delta t$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta P_L}{\Delta t} = \frac{dP_L}{dt} = P_L - (R + E_{LL} + E_{LS}) \rightarrow \text{at equilibrium: } \frac{dP_L}{dt} = 0$$

$$P_L = R + E_{LL} + E_{LS}$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta P_S}{\Delta t} = \frac{dP_S}{dt} = P_S + R - (E_{SS} + E_{SL}) \rightarrow \text{at equilibrium: } P_S + R = E_{SS} + E_{SL}$$

From the above calculations, we derive the two equations at steady state:

	<b><u>input</u></b>	=	<b><u>output</u></b>
1. Conservation of water in the sea:	$P_S + R$	=	$E_{SS} + E_{SL}$
2. Conservation of water on the land:	$P_L$	=	$R + E_{LL} + E_{LS}$

Where:  $P_L = E_{LL} + E_{SL}$ ,  $P_S = E_{LS} + E_{SS}$

After 20% reduction of evapotranspiration from the land due to deforestation, let the primed variables be the new steady-state condition:

$$E'_{LL} = 0.8E_{LL}$$

$$E'_{LS} = 0.8E_{LS}$$

$$E'_{SS} = E_{SS}$$

$$E'_{SL} = E_{SL}$$

$$R' = E'_{SS} + E'_{SL} - P'_S = E'_{SS} + E'_{SL} - E'_{LS} - E'_{SS}$$

$$= E'_{SL} - E'_{LS} = (61.5 - (0.8 * 15.5)) \times 10^3 \text{ km}^3/\text{yr} \quad \text{(from equation 1)}$$

$$= 49.1 \times 10^3 \text{ km}^3/\text{yr}$$

$$\begin{aligned}
P'_L &= R' + E'_{LL} + E'_{LS} \\
&= R' + 0.8E_{LL} + 0.8E_{LS} \\
&= 49.1 \times 10^3 \text{ km}^3/\text{yr} + 0.8 \times 46.5 \times 10^3 \text{ km}^3/\text{yr} + 0.8 \times 15.5 \times 10^3 \text{ km}^3/\text{yr} \quad (\text{from equation 2}) \\
&= (49.1 \times 10^3 + 37.2 \times 10^3 + 12.4 \times 10^3) \text{ km}^3/\text{yr} \\
&= 98.7 \times 10^3 \text{ km}^3/\text{yr}
\end{aligned}$$

$R'$  is 6.7% higher than  $R$ ;  $P'_L$  is 8.6% lower than  $P_L$ . This means that deforestation will lead to an increased runoff and reduced precipitation. Increased runoff leads to soil erosion which leads to river and lake pollution. Both increased runoff and reduced precipitation are conducive conditions toward desertification.

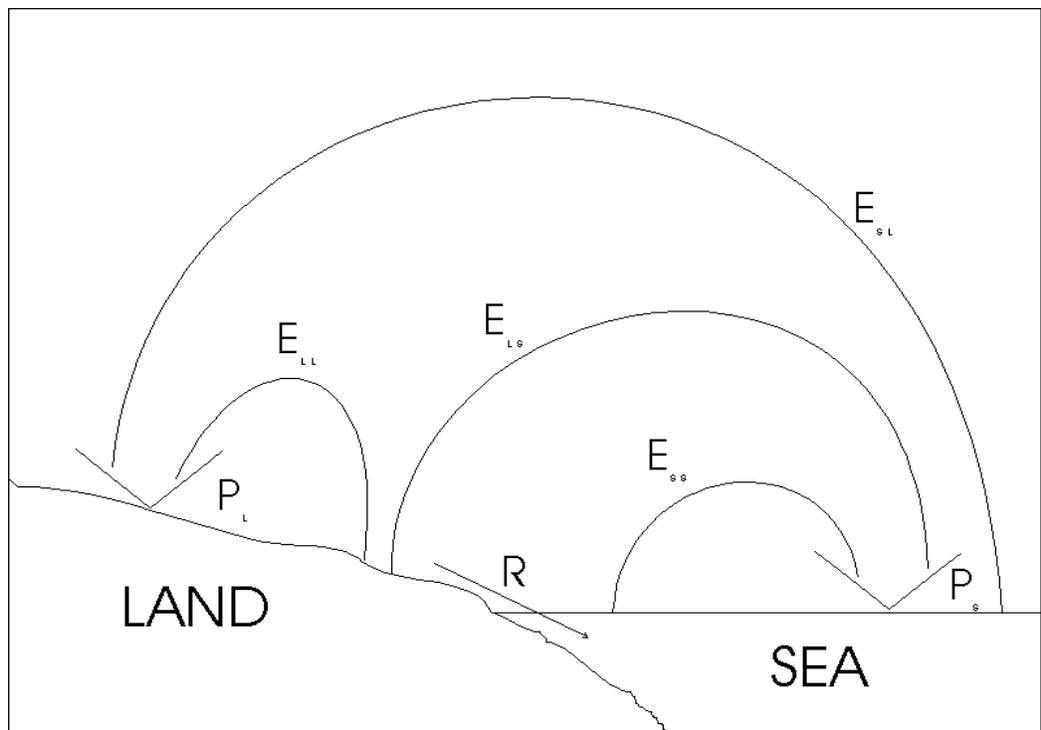


Figure 5.7. A schematic diagram of a hydrologic cycle.

✍ Example 5.8. Biomagnification of Trace Substances, a Biomedical Application

The process of biomagnification refers to the tendency of certain toxic substances to concentrate in organisms, working their way up in the various hierarchies of the food chain as larger organisms consume smaller ones. One source of toxic substances is pesticide residues. Pesticides are chemicals designed to eliminate weeds, insects, and other destructive organisms. Its application is common in lawn care, golf courses, and agricultural production. Some of the chemical compounds in the pesticides do not break down easily. In the course of time, these chemicals end up in rivers and lakes through sediment runoff or atmospheric deposition. Once in lakes and streams, these chemicals get ingested by phytoplanktons, which

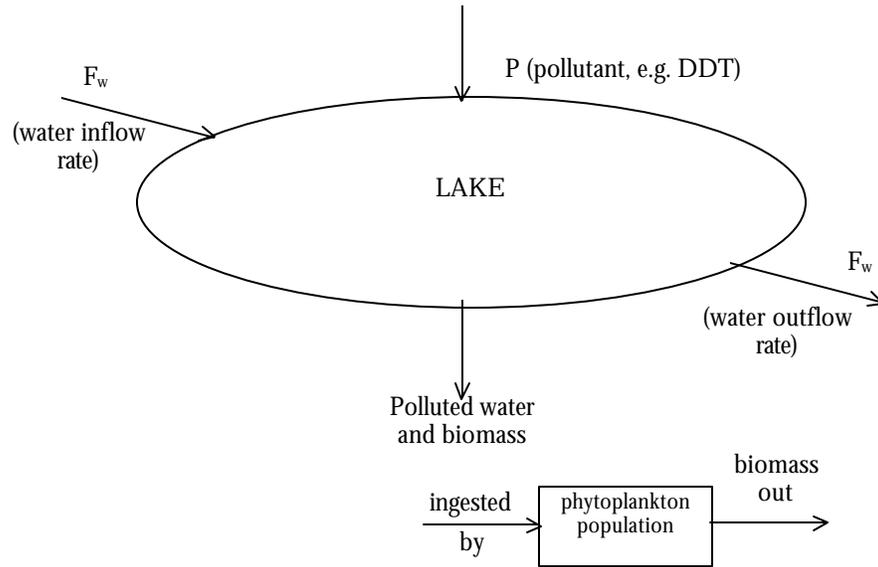
are then eaten by the fish. Fish also accumulate these toxic substances from water passing over their gills. Then, terrestrial animals, birds, and human beings eat the contaminated fish, and so up in the food chain.

A major national concern at present is the accumulation of DDT (DichloroDiphenylTrichloroethane) in the environment. A U.S. Geological Survey circular (Rinella, et al., 1993) reported the persistence of DDT pesticide in the Yakima River Basin, which is located on the eastern slope of the Cascade Range in south-central Washington. The broad toxicity of DDT and its breakdown products, DDE (DichloroDiphenyldichloroEthylene) and DDD (DichloroDiphenylDichloroethane), can affect many organisms, such as fish and animals, other than the insects for which it was designed. Its persistence in the environment can lead to dangerous accumulations and adversely affect the reproductive capabilities of birds and other wildlife. Its cancer-causing potential can possibly affect human health.

In this exercise, we shall try to understand how biomagnification of DDT occurs (Harte, 1985, pp. 205-210). We shall try also to derive a formula for the concentration of a trace substance in each link in the food chain. There are several factors that determine the degree of bioconcentration of a trace substance (Harte, 1988). These factors are:

1. Retention factor—the amount of ingested trace substance retained in the tissues rather than excreted or metabolized.
2. Incorporation efficiency—The ratio of weight gain to the weight (amount) of food consumed. If an animal is very inefficient in building new tissue out of its food source, it must consume a lot of food to grow by any specified amount. That larger amount of ingested food is accompanied by a larger amount of trace substance. Thus, for a given retention factor, the lower the incorporation efficiency, the greater the rate of bioconcentration. In ecological literatures, incorporation efficiency is about 10%, that is, an animal would gain one gram of weight for every 10 grams of food consumed. The remaining 90% of the food is excreted or metabolized.
3. Relative growth rate—For a given retention factor and incorporation efficiency, the more weight an animal puts on during its lifetime, compared to its weight at birth, the greater is the percentage increase in the concentration of a trace substance over the lifetime of the organism.
4. Location in the food chain—The fish bioconcentrates the trace substance from the planktons it eats, and if other factors are equal, accumulates a higher concentration than that in the plankton. The higher in a food chain an organism feeds, the greater is the concentration effect for that organism.
5. Environmental contamination—The contamination of soil and water initiates the food-chain effect and its amount is therefore an important determinant of the ultimate concentration in all organisms.

**Solution:**



- The rate of change of phytoplankton biomass ( $B$ ) is:

$$\Delta B = (\text{inflow rate} - \text{outflow rate})\Delta t$$

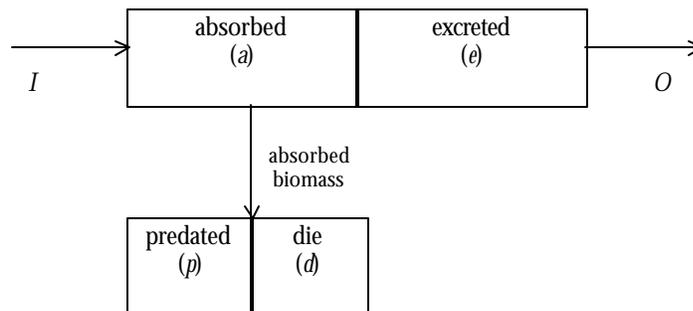
$$\left. \begin{aligned} \frac{\Delta B}{\Delta t} &= I - O \\ \lim_{\Delta t \rightarrow 0} \frac{\Delta B}{\Delta t} &= \frac{dB}{dt} = I - O \end{aligned} \right\} \text{ where } \begin{cases} I = \text{biomass intake per day} \\ O = \text{biomass out per day} \end{cases}$$

At steady state:  $\frac{dB}{dt} = 0 \Rightarrow I = O$

$\Rightarrow$  **Equation 1:**  $I = O$  at steady state.

**FACTORS AFFECTING BIOMAGNIFICATION**

**Factor 1: Retention**



$\Rightarrow$  **Equation 2:**  $I = a + e$

⇒ **Equation 3:**  $a = p + d$

⇒ **Equation 4:**  $O = e + p + d$

**Factor 2: Incorporation Efficiency (E)**

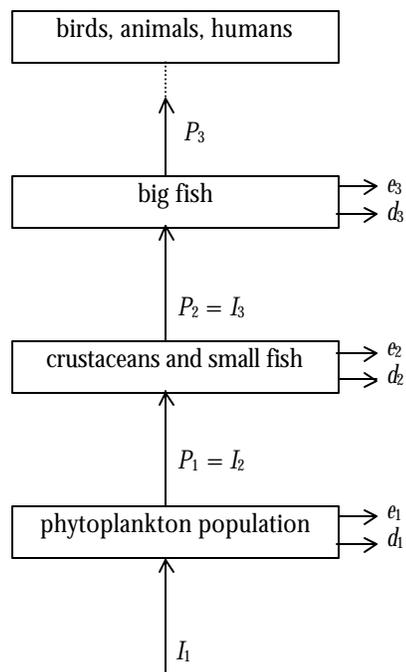
$$E = \frac{a}{I} = \frac{p+d}{I}$$

⇒ **Equation 5:**  $E = \frac{p+d}{I}$

**Factor 3: Relative growth rate**

more body weight ® more food intake ® higher accumulation of pollutant

**Factor 4: Location in the food chain**



⇒ **Equation 6:**  $p_1 = I_2$

✓ *Question:* Are you seeing a pattern for moving from the lower level component (phytoplankton) to the higher level (big fish)?

- The generalized equations for biomass flow are:

⇒ **Eq. 1:**  $I_i = \theta_i$

⇒ **Eq. 2:**  $I_i = a_i + e_i$

⇒ **Eq. 3:**  $a_i = p_i + d_i$

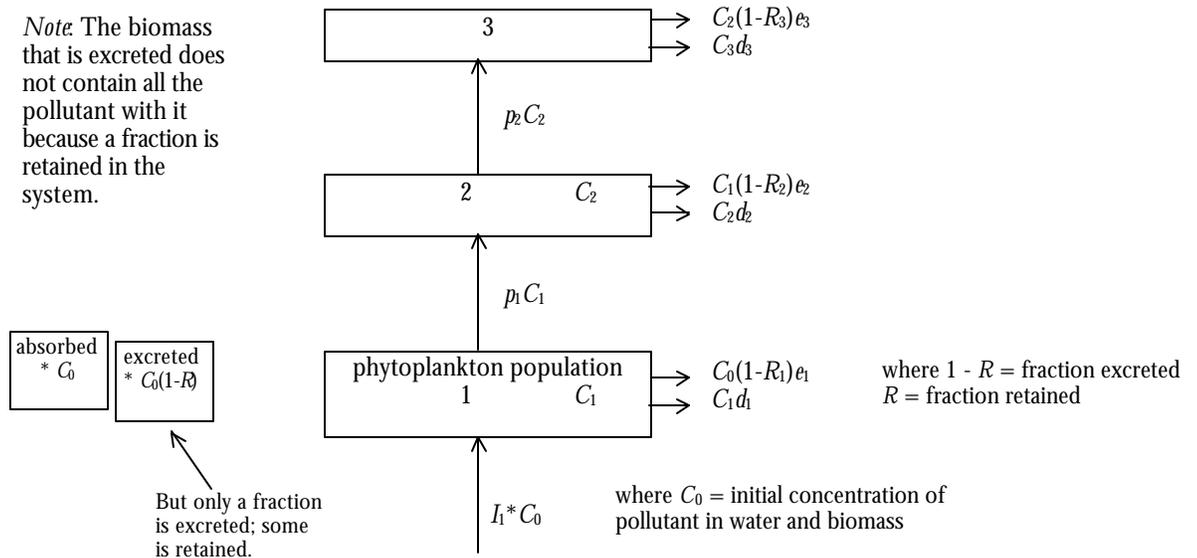
⇒ **Eq. 4:**  $\theta_i = e_i + p_i + d_i$

⇒ **Eq. 5:**  $E_i = \frac{p_i + d_i}{I_i} = \frac{a_i}{I_i}$

⇒ **Eq. 6:**  $p_i = I_{i+1}$

**Factor 5: Degree of Environmental Contamination**

- Now, what about the concentration of the pollutant in the biomass flow?



**System 1:** At steady state, inflow rate = outflow rate

$$I_1 C_0 = C_1 d_1 + C_0(1 - R_1)e_1 + p_1 C_1$$

- From Eq. 6,  $I_1 = p_0$ , so we have:

$$p_0 C_0 = C_1 d_1 + C_0(1 - R_1)e_1 + p_1 C_1$$

**System 2:**

$$p_1 C_1 = C_2 d_2 + C_1(1 - R_2)e_2 + p_2 C_2$$

- Now, we can generalize the equations at all system levels into:

$$\begin{aligned} p_{i-1}C_{i-1} &= C_i d_i + C_{i-1} e_i (1 - R_i) + p_i C_i \\ &= C_i (p_i + d_i) + C_{i-1} e_i (1 - R_i) \end{aligned}$$

- To solve for  $C_i$  at any  $i$  level and express it in terms of  $R$  and  $E$ :

$$\begin{aligned} C_i (p_i + d_i) &= p_{i-1} C_{i-1} - C_{i-1} e_i (1 - R_i) \\ &= C_{i-1} (p_{i-1} - e_i (1 - R_i)) \\ C_i &= \frac{C_{i-1} [p_{i-1} - e_i (1 - R_i)]}{p_i + d_i} \end{aligned}$$

- From Eq. 5:  $I_i E_i = p_i + d_i$

$$C_i = \frac{C_{i-1} [p_{i-1} - e_i (1 - R_i)]}{E_i I_i}$$

- From Eq. 6:  $p_{i-1} = I_i$

$$C_i = \frac{C_{i-1} [I_i - e_i (1 - R_i)]}{E_i I_i}$$

- From Eq. 2:  $I_i = a_i + e_i$

$$C_i = \frac{C_{i-1} [I_i - (I_i - a_i)(1 - R_i)]}{E_i I_i}$$

- From Eq. 5:  $E_i I_i = a_i$

$$\begin{aligned} C_i &= \frac{C_{i-1} [I_i - (I_i - E_i I_i)(1 - R_i)]}{E_i I_i} \\ &= \frac{C_{i-1} [I_i - (I_i - I_i R_i - E_i I_i + E_i I_i R_i)]}{E_i I_i} \\ &= \frac{C_{i-1} [I_i - I_i + I_i R_i + E_i I_i - E_i I_i R_i]}{E_i I_i} \\ &= \frac{C_{i-1} I_i [R_i + E_i - E_i R_i]}{E_i I_i} = \frac{C_{i-1} (R_i + E_i - E_i R_i)}{E_i} \\ &= C_{i-1} \left[ \frac{E_i}{E_i} + \frac{R_i}{E_i} - \frac{E_i}{E_i} R_i \right] \\ C_i &= C_{i-1} \left[ 1 + \frac{R_i (1 - E_i)}{E_i} \right] \end{aligned}$$

- *Final Solution:*  $C_i = C_{i-1} \left[ 1 + R_i \frac{(1 - E_i)}{E_i} \right]$

- Therefore, the concentration of the pollutant in each population level is:

$$C_1 = C_0 \left[ 1 + R_1 \frac{(1 - E_1)}{E_1} \right]$$

$$C_2 = C_1 \left[ 1 + R_2 \frac{(1 - E_2)}{E_2} \right]$$

$$= C_0 \left[ 1 + R_1 \frac{(1 - E_1)}{E_1} \right] \left[ 1 + R_2 \frac{(1 - E_2)}{E_2} \right]$$

✓ Show that:

$$p_{i-1}C_{i-1} = C_i d_i + C_i p_i + C_{i-1} e_i (1 - R_i)$$

would lead to:

$$C_i = C_{i-1} \left[ 1 + R_i \frac{(1 - E_i)}{E_i} \right].$$

- Shorter way to derive the equation (*Note:*  $p_{i-1} - e_i = I_i - e_i \Rightarrow I_i - e_i = a_i$ ):

$$p_{i-1}C_{i-1} = C_i d_i + C_i p_i + C_{i-1} e_i (1 - R_i)$$

$$= C_i (d_i + p_i) + C_{i-1} e_i (1 - R_i)$$

$$C_i = \frac{-C_{i-1} e_i (1 - R_i) + p_{i-1} C_{i-1}}{(d_i + p_i)} = \frac{C_{i-1} [p_{i-1} - e_i (1 - R_i)]}{a_i}$$

$$C_i = \frac{C_{i-1} [p_{i-1} - e_i + e_i R_i]}{a_i}$$

$$= C_{i-1} \frac{[a_i + e_i R_i]}{a_i} = C_{i-1} \left[ 1 + \frac{e_i R_i}{a_i} \right]$$

$$= C_{i-1} \left[ 1 + \frac{(I_i - a_i) R_i}{E_i I_i} \right] = C_{i-1} \left[ 1 + R_i \frac{(I_i - E_i I_i)}{I_i} \right]$$

$$= C_{i-1} \left[ 1 + R_i \frac{I_i (1 - E_i)}{E_i I_i} \right] = C_{i-1} \left[ 1 + R_i \frac{(1 - E_i)}{E_i} \right]$$

$$C_i = C_{i-1} \left[ 1 + R_i \frac{(1 - E_i)}{E_i} \right]$$

## Exercises

1. Consider a problem where 50 kg of a pollutant P is dumped daily into a lake. Assume that the pollutant is stable, highly soluble, and uniformly mixed in the lake. The lake volume is given to be  $4 \times 10^7 \text{ m}^3$  and the average water flow-through rate is  $5 \times 10^4 \text{ m}^3 \text{ d}^{-1}$ . What is the pollutant concentration over time?

2. From Example 5.2, show the relationship of  $\frac{C_2(t)}{C_1(0)}$  for
 
$$\begin{cases} V_1 = 10V_2, & V_1 > V_2 \\ V_1 = V_2 \\ V_2 = 10V_1, & V_2 > V_1 \end{cases}$$

3. *Carbon in the atmosphere.* Carbon is a major chemical element in the biosphere (Harte, 1985, p.28-31). It can be found in living biomasses or in the atmosphere as  $\text{CO}_2$ . An increase in atmospheric  $\text{CO}_2$  has been associated with the “greenhouse” global-warming effect.

Given the following data:

C in living continental organisms:  $5.6 \times 10^{14} \text{ kg C}$

C flow from terrestrial organic matter:  $5 \times 10^{13} \text{ kg C yr}^{-1}$

C in living marine organisms:  $2 \times 10^{12} \text{ kg C}$

C flow from oceanic organic matter:  $2.5 \times 10^{13} \text{ kg C yr}^{-1}$

What are the residence times of carbon in continental and marine vegetation?

4. *Atmospheric pollution.* Ethane ( $\text{C}_2\text{H}_6$ ) is a constituent of natural gas that is emitted to the atmosphere as pollutant whenever natural gas escapes unburned at wells and other sources (Harte, 1985, pp. 40-44). Given the following data:

$C_N$  (average concentration of ethane in the troposphere of the northern hemisphere) = 1.0 ppb(v)

$C_S$  (average concentration of ethane in the troposphere of the southern hemisphere) = 0.5 ppb(v)

Assuming that the total exit rate from each hemisphere’s troposphere is proportional to the concentration in the respective troposphere, and knowing that 3 percent as much natural gas escapes to the atmosphere unburned, estimate the net rate of ethane flow across the equator.

Available data to help solve the problem:

- In 1980, gas was burned at a rate of  $6 \times 10^{19} \text{ J yr}^{-1}$
  - Energy content of natural gas:  $4 \times 10^7 \text{ J m}^3$  (STP)
  - 1 mole of gas (STP) occupies 22.4 li
  - There are 44.6 moles of gas per  $1 \text{ m}^3$
  - 6% of natural gas is ethane
  - The atmosphere has  $1.8 \times 10^{20}$  moles of air
  - Assume all natural gas is mined and vented in the northern hemisphere
5. *Freeze or radon?* Suppose a person is contemplating to build a house. The style and other features of the building will vary depending on whether he/she lives in Michigan or in California, or if he/she has access to information about certain health side-effects of different designs. Radioactive radon gas ( $\text{Rn}_{222}$ ) is produced in the earth's crust and is one of the main natural sources of ionizing radiation (Archer et al., 1987, p. 151). It is carried to the surface by diffusion and exhalation of volatile isotopes from porous rocks and soils into water and the atmosphere, by emission from volcanoes and related phenomena, by weathering, and through leaching by groundwater (Archer et al., 1987, p. 151). Suppose  $\text{Rn}_{222}$  enters a building at the rate of one picocurie  $\text{s}^{-1}$  for every  $\text{m}^2$  of foundation area (Harte, 1985, pp. 59-64). Consider a house with a foundation area of  $200 \text{ m}^2$  and an air volume of  $1000 \text{ m}^3$ . Assume that the house is well designed for energy conservation so that the ventilation rate is low with only 10 percent of the air in the house is exchanged with outdoor air every hour.
- (a) What will be the average steady-state concentration of  $\text{Rn}_{222}$  in the house?
- (b) In the steady state, what whole-body radiation dose, in  $\text{rads yr}^{-1}$ , will an adult male receive directly from  $\text{Rn}_{222}$  decay if he spends 12 hr a day in the house and derives the entire dose from the decay of radon in the inhaled air in his lungs?
6. *Acidification of the Atmosphere.* Sulfur dioxide ( $\text{SO}_2$ ) is present in the atmosphere from natural and anthropogenic sources. The sulfur cycle can be altered by emissions from combustion processes of coal, which contains an average of 2 percent sulfur.  $\text{SO}_2$  that is emitted into the atmosphere is oxidized to become sulphates. High concentrations of  $\text{SO}_2$  in the air leads to acid rain which can destroy forests, lakes, and food production systems. Without combustion emissions, natural sources add  $\text{SO}_2$  to the atmosphere at a rate of about  $10^8$  tons S/yr (Harte, 1985, pp. 32-33). The background concentration of atmospheric  $\text{SO}_2$ , measured in remote areas where anthropogenic sources are

not likely to have much influence, is about 0.2 parts per billion, by volume [ppb(v)]. What is the residence time of atmospheric SO<sub>2</sub> in the remote regions? With combustion emissions, what is the globally averaged SO<sub>2</sub> concentration in the atmosphere Harte, 1985, pp. 34-35)? What is the SO<sub>2</sub> concentration in industrialized regions like the northeastern United States?

Hints in solving the problem:

1. Information given from the problem:

- SO<sub>2</sub> concentration = 0.2 ppb(v) =  $0.2 \times 10^{-9}$  unit volume of SO<sub>2</sub> per unit volume of air =  $0.2 \times 10^{-9}$  moles of SO<sub>2</sub> / mole of air
- Inflow-outflow rate of SO<sub>2</sub> in the atmosphere from natural sources =  $10^8$  t SO<sub>2</sub> yr<sup>-1</sup>

2. Additional information:

- 1 mole of gas occupies 22.4 liters at STP
- Atmosphere has  $1.8 \times 10^{20}$  moles of air
- Atomic weight of: Sulfur (S) is 32 g and oxygen (O) is 16 g
- In 1980, anthropogenic sulfur emissions to the atmosphere is  $8.5 \times 10^7$  t SO<sub>2</sub> yr<sup>-1</sup>
- Northeastern United States occupies only 0.2% of the Earth's surface area but emits about  $10^7$  t of SO<sub>2</sub> yr<sup>-1</sup> or 12% of the global anthropogenic SO<sub>2</sub>.

3. Part I, solve for the residence time, T<sub>SO<sub>2</sub></sub>, from natural sources:

- $T_{\text{SO}_2} = \text{mass of SO}_2 \text{ in the atmosphere} / \text{rate of flow of SO}_2 \text{ in the atmosphere}$

4. Part II, solve for global and regional average SO<sub>2</sub> concentration, using 1980 data on anthropogenic sulfur emissions:

- Calculate mass of global SO<sub>2</sub> emissions (natural and anthropogenic sources) in the atmosphere.

- Calculate for global SO<sub>2</sub> concentration, expressed in ppb (moles of SO<sub>2</sub> per mole of air).
- Solve for regional (Northeastern U.S.) SO<sub>2</sub> concentration, expressed in ppb.

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- Harte, J. 1985. *Consider a Spherical Cow*. W. Kaufmann, Inc., California.
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## Oscillations and Stability in Biological Systems

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### PRINCIPLE

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Biological systems oscillate.

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### TOPICS

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Simple harmonic motion

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Damped motion

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Damped forced vibrations

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Forced free vibrations

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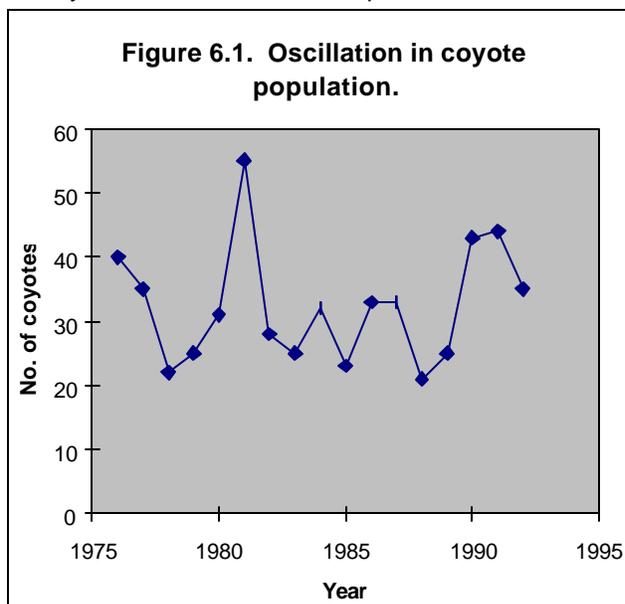
Stability test by the isocline and phase-plane methods

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Oscillation is a typical characteristic of biological systems. It is a phenomenon that occurs at all levels of the hierarchy in society, from the cell all the way up to the universe. The growth rate function can be separated into two components: growth rate due to the natural environmental conditions, such as weather and food limitation, and growth rate due to birth-and-death processes and overlapping generations. It is the latter type of growth rate that exhibits oscillation in population dynamics. Table 6.1 and Fig. 6.1 show the oscillation in the coyote population (Minnesota Dept. of Natural Resources, 1993).

Table 7.1. Number of coyotes that visited a scent post.

Year	Coyote
1976	40
1977	35
1978	22
1979	25
1980	31
1981	55
1982	28
1983	25
1984	32
1985	23
1986	33
1987	33
1988	21
1989	25
1990	43
1991	44
1992	35



In order to understand the oscillatory phenomenon, this chapter introduces Newton's law of motion in describing a simple pendulum. Newton's law provides a good framework for modeling back-and-forth or up-and-down motion observed in biological systems. A simple pendulum consists of small bob with a constant mass  $m$  attached to a slim, rigid, and massless string of length  $l$ , which is itself connected to a pivot  $P$ , moving in a vertical circular arc (Weidner and Sells, 1965). The straight down position finds the bob at rest, where the angle  $\theta$  is zero. The swinging motion can be initiated by pulling the bob to the side and releasing it. When the pendulum is displaced from rest, the only restoring force is that of gravity, which tends to pull it back downward (Beltrami, 1987).

The radial and tangential forces acting on the body, the tension  $T$  and the weight  $mg$ , are shown at the moment when the string makes an angle  $\theta$  with respect to the vertical and the body is descending with instantaneous speed along the circular path (Weidner and Sells, 1965). The radial forces are forces along the direction of the string and the tangential forces are forces along the path. Since the motion is confined to lie along the circular arc in the plane of radius  $l$  about  $P$ , we are only interested in that component of gravitational force that is tangent to the arc. One can allow viscous damping and this too is taken to be tangential to the arc (Beltrami, 1987). Figure 6.2 shows a force diagram of a simple pendulum. The restoring force along the arc due to gravity is  $mg \sin \theta$ . The body accelerates under the influence of force; otherwise it is either at rest or in motion at a uniform velocity. If the line of motion is the  $x$  axis, then at any time  $t$ ,  $x(t)$  is the position,  $x'(t)$  is the velocity, and  $x''(t)$  is the acceleration. If  $F$  is the sum of the forces acting on the body, its motion satisfies the differential equation

$$F = mx'' \tag{6.1}$$

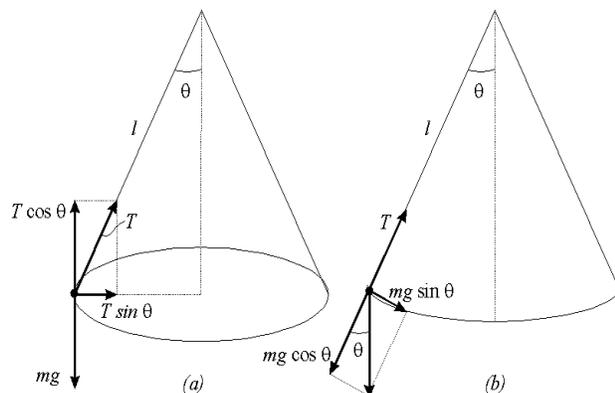


Figure 6.2. A force diagram of a simple pendulum.

Equation (6.1) is known as Newton's law of motion. Our main interest in this study is to understand something about where its mass will be found in the

long run rather than on the specifics of how it wanders about in the short term. Will the displaced bob eventually return to its equilibrium state? In the case of biological species, will the oscillation in the population eventually stop and the population settle to a steady state?

In order to understand the oscillation dynamics, we shall apply the law of motion to a swinging human leg.

## 6.1. Simple harmonic motion

Newton's law of motion will be applied in modeling a free-swinging human leg, as illustrated by Eisen (1988). The leg has a mass of 11.2 kg at the center of gravity of the leg and is 90 cm long. The swinging motion can be initiated by raising the leg to the side and releasing it. The moment of inertia of the thin cylindrical leg swinging about the hip joint is given as

$$J = \frac{mL^2}{3} \quad (6.2)$$

where  $L$  is the length of the leg. If we replace force with torque,  $T$ , Newton's law becomes

$$T = J\theta'' \quad (6.3)$$

The torque perpendicular to the leg can be defined as the product of force and distance that the force acts from the center of rotation. It is obtained by resolving the force due to the weight of the leg,  $mg$ , into a component perpendicular to the leg. Thus,

$$T = -mg \cos(90 - \theta) \left(\frac{L}{2}\right) = -\frac{mgL \sin \theta}{2} \quad (6.4)$$

Substituting the values of  $J$  and  $T$  from (6.2) and (6.4), respectively, into (6.3) gives

$$\mathbf{q}'' + \frac{3g \sin \mathbf{q}}{2L} = 0. \quad (6.5)$$

Using the approximation  $\sin \mathbf{q} = \mathbf{q}$  and letting  $\mathbf{v} = \left(\frac{3g}{2L}\right)^{1/2}$ , (6.5) becomes

$$\mathbf{q}'' + \mathbf{v}^2 \mathbf{q} = 0 \quad (6.6)$$

where  $\mathbf{v}$  is the undamped natural frequency. Using the DSolve function of Mathematica, the general solution to (6.6) is

```
DSolve[ {theta''[t] + w^2*theta[t] == 0}, theta[t], t]
```

$$\mathbf{q} = c_2 \cos \mathbf{v}t + c_1 \sin \mathbf{v}t \quad (6.7)$$

The value of  $\mathbf{v}$  can be calculated by substituting the standard value of the gravitational acceleration,  $g$ , with  $980.665 \text{ cm s}^{-2}$  and  $L$  with  $90 \text{ cm}$ , resulting in  $\mathbf{v}$  having a value of approximately  $4 \text{ radians s}^{-1}$  (Eisen, 1988). Since  $\mathbf{v}$  is real, (6.7) becomes

```
w = 4;
DSolve[ {theta''[t] + w^2*theta[t] == 0}, theta[t], t]
```

$$\mathbf{q} = c_2 \cos 4t - c_1 \sin 4t \quad (6.8)$$

which describes a periodic motion. If we set the initial position to  $\mathbf{q}_0$ , where  $\mathbf{q}_0$  is not zero since the body is initially stretched or compressed, and the velocity set to zero, since the body has no velocity when it is first let go (Beltrami, 1987), that is,  $\mathbf{q}(0) = \mathbf{q}_0$ ,  $\mathbf{q}'(0) = 0$ , then the solution, using the DSolve function of Mathematica, becomes a cosine term

```
DSolve[{theta''[t] + w^2*theta[t] == 0, theta'[0]==0,
theta[0]==0.5}, theta[t], t] //N
```

$$\mathbf{q}(t) = \mathbf{q}_0 \cos 4t \quad (6.9)$$

The amplitude (the distance the body swings either to the right or to the left of the rest position) is represented by  $\mathbf{q}_0$ , the period  $T$  (the time for one complete cycle of the mass to oscillate back and forth about the zero position) is calculated from  $T = 2\mathbf{p} / \mathbf{v}$ , and the frequency of oscillation,  $\mathbf{v}$ , is the number of cycles which are completed during the interval  $2\mathbf{p}$  (Beltrami, 1987). As the period  $T$  decreases, the frequency  $\mathbf{v}$  increases. Eisen (1988) used (6.6) to the model the speed of walking. Assuming a 8-ft pace, he calculated the walking speed from  $\mathbf{v}/2\mathbf{p}$  to be 0.64 double-paces/sec or 2.6 mph. If we use the NDSolve and Plot functions of Mathematica and give the initial conditions  $\mathbf{q}(0) = 0.5$ ,  $\mathbf{q}'(0) = 0$ , we can see from Fig. 6.3 that (6.6) describes a simple harmonic motion of a free-swinging leg.

```
w = 4;
eq3 = NDSolve[{theta''[t] + w^2*theta[t] == 0, theta'[0]==0,
theta[0]==0.5}, theta[t], {t,0,4*Pi}];
Figure1 = Plot[Evaluate [theta[t] /. eq3], {t,0,4 Pi},
PlotRange->{-0.5,0.5}]
```

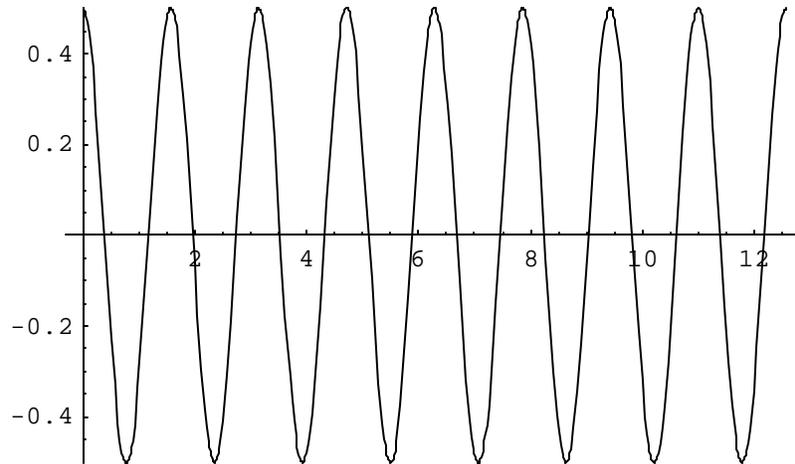


Figure 6.3. Simple harmonic motion of a free swinging leg.

## 6.2 Damped motion

Eisen (1988) extended the illustration of Newton's law to a free-swinging leg with damping or resistance. Eisen indicated that although the energy dissipation effects are small in relaxed walking, they nevertheless exist. Energy dissipation effects are represented by adding a torque to the right-hand side of (6.6), such as  $\mathbf{q}'' + \mathbf{v}^2 \mathbf{q} = -r \mathbf{q}'$ , with a negative sign added to represent viscous resistance of the tissues opposing the motion (Eisen, 1988), which translates to

$$\mathbf{q}'' + r \mathbf{q}' + \mathbf{v}^2 \mathbf{q} = 0 \quad (6.10)$$

where  $r > 0$ . The motion demonstrated by (6.10) will depend on the strength of the damping force. If  $r > 2\omega$ , the motion is overdamped, the trajectory  $\mathbf{q}(t) \rightarrow 0$  as  $t \rightarrow \infty$ , and so the amplitude of the oscillations decreases over time. The leg motion is no longer periodic and eventually ceases to move after a long period of time, as shown in Fig. 6.4.

```
r = 20;
eq4 = NDSolve[{theta''[t] + r*theta'[t] + w^2*theta[t] == 0,
theta'[0]==0, theta[0]==0.5}, theta[t], {t,0,10}];
Figure2 = Plot[Evaluate [theta[t] /. eq4], {t,0,10},
PlotRange->{0,0.5}]
```

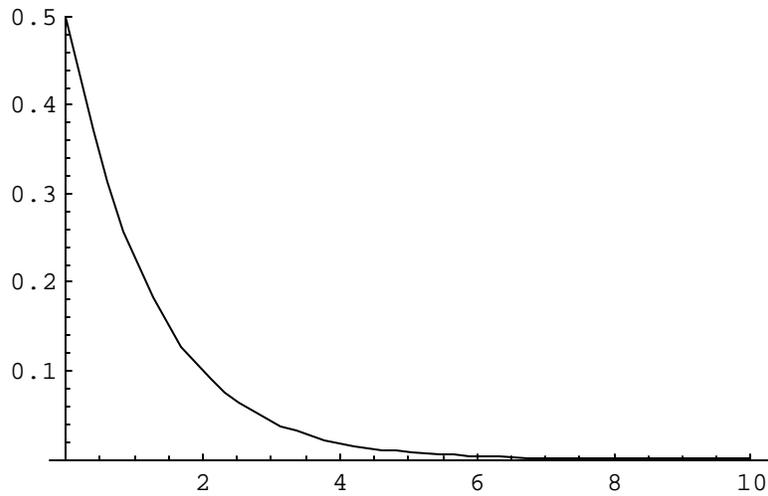


Figure 6.4. An overdamped motion, where  $r > 2\omega$ .

If  $r = 2\omega$ , the motion is critically damped, the trajectory  $\mathbf{q}(t) \rightarrow 0$ , as  $t \rightarrow \infty$ , and the leg eventually ceases to move without swinging past the origin, as shown in Fig. 6.5.

```
r = 8;
eq5 = NDSolve[{theta''[t] + r*theta'[t] + w^2*theta[t]==0,
theta'[0]==0, theta[0]==0.5}, theta[t], {t,0,4*Pi}];
Figure3 = Plot[Evaluate [theta[t] /. eq5], {t,0,4*Pi},
PlotRange->{0,0.5}]
```

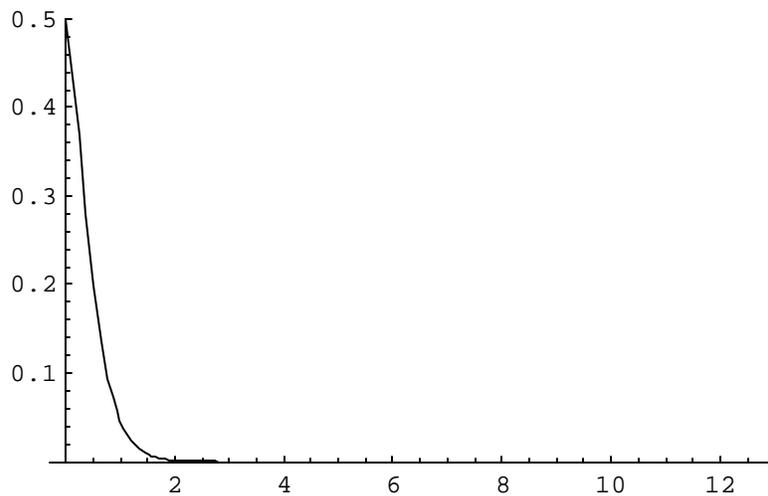


Figure 6.5. A critically damped motion, where  $r = 2\omega$ .

If  $r < 2\omega$ , the motion is underdamped and the trajectory  $\mathbf{q}(t)$  oscillates back and forth along the rest position with decreasing amplitude. After a long period of time, the leg ceases to move (Fig. 6.6).

```
r = 1;
eq6 = NDSolve[{theta''[t] + r*theta'[t] + w^2*theta[t] == 0,
  theta'[0]==0, theta[0]==0.5}, theta[t], {t,0,4*Pi}];
Figure4 = Plot[Evaluate [theta[t] /. eq6], {t,0,4*Pi},
PlotRange->{{0,12},{-0.25,0.25}}, AxesLabel->{"Time",
"Theta"}]
```

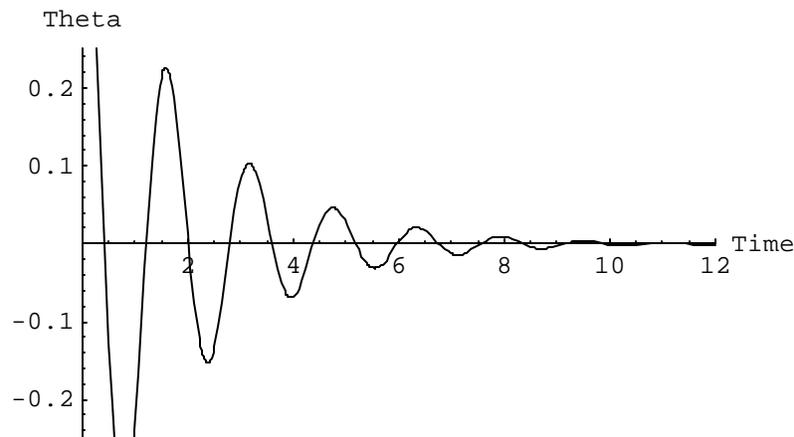


Figure 6.6. An underdamped motion, where  $r < 2\omega$ .

### 6.3 Damped forced vibrations

Eisen (1988) further illustrated a situation where a time-varying muscle torque of the form  $T_m = T_0 \cos j t$  is added to (6.10) as a driving force, such as:

$$\mathbf{q}'' + r\mathbf{q}' + \mathbf{v}^2\mathbf{q} = T_0 \cos j t \quad (6.11)$$

$T_m$  is called the input or forcing function and the trajectory of  $T_m$  is shown in Fig. 6.7.

```
phi = 2;
To=1;
Figure5 = Plot[To*Cos[phi*t], {t,0,4*Pi}, AxesLabel->{"time", "Tm"}]
```

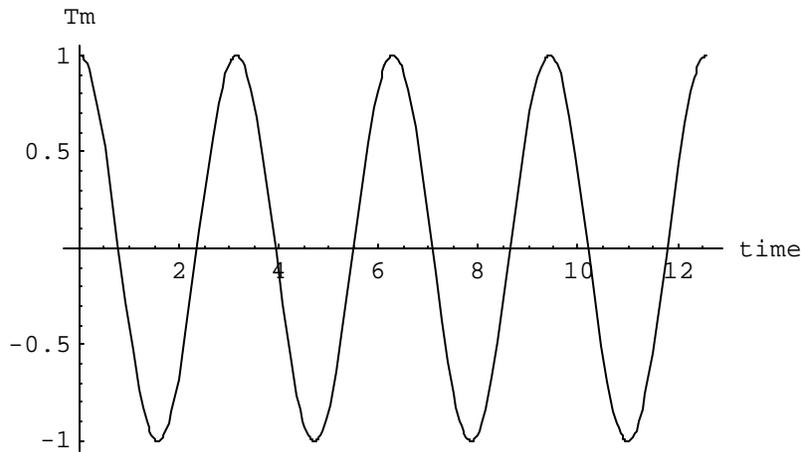


Figure 6.7. Trajectory of the forcing function  $T_m$ .

We have seen the trajectory of  $\mathbf{q}(t) \rightarrow 0$  as  $t \rightarrow \infty$  in the unforced leg (Fig. 6.4 through Fig. 6.6), where the motion eventually ceases. However, with the forcing function added, the trajectory of  $\mathbf{q}(t)$  over time, as  $t \rightarrow \infty$ , follows the frequency of the forcing function, as shown in Fig. 6.8.

```

r = 8;
eq7 = NDSolve[{theta''[t] + r*theta'[t] + w^2*theta[t] ==
To*Cos[phi*t],
  theta'[0]==0, theta[0]==0.5}, theta[t], {t,0,10},
MaxSteps->2000];
Figure6 = Plot[Evaluate [theta[t] /. eq7], {t,0,10}]

```

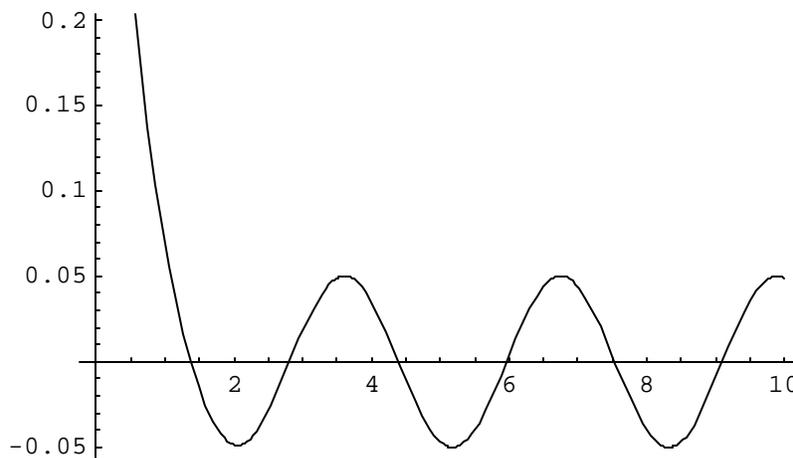


Figure 6.8. Trajectory of  $\theta(t)$  with forcing function  $T_m$ .

## 6.4 Forced free vibrations

Eisen (1988) illustrated a situation where damping is zero but a forcing function is present. Equation (6.11) becomes

$$\mathbf{q}'' + \mathbf{v}^2 \mathbf{q} = T_0 \cos \mathbf{j} t \quad (6.12)$$

The trajectory of  $\mathbf{q}(t)$  as  $t \rightarrow \infty$  is shown in Fig. 6.9.

```
r = 0;
eq10 = NDSolve[{theta''[t] + r*theta'[t] + w^2*theta[t] ==
To*Cos[phi*t],
  theta'[0]==0, theta[0]==0.5}, theta[t], {t,0,10}];
Figure7 = Plot[Evaluate [theta[t] /. eq10], {t,0,10}]
```

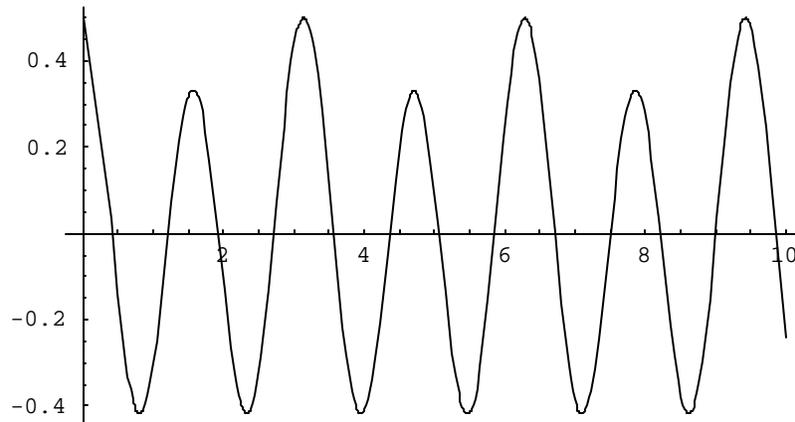


Figure 6.9. Trajectory of  $\theta(t)$  with zero damping but with forcing function  $T_m$ .

When the frequency of the external force  $\mathbf{j}$  is equal to the natural frequency of the system  $\mathbf{v}$ , the trajectory oscillates with increasing amplitude, as shown in Fig. 6.10.

```
r = 0;
phi = w;
eq11 = NDSolve[{theta''[t] + r*theta'[t] + w^2*theta[t] ==
To*Cos[phi*t],
  theta'[0]==0, theta[0]==0.5}, theta[t], {t,0,10}];
Figure8 = Plot[Evaluate [theta[t] /. eq11], {t,0,10}]
```

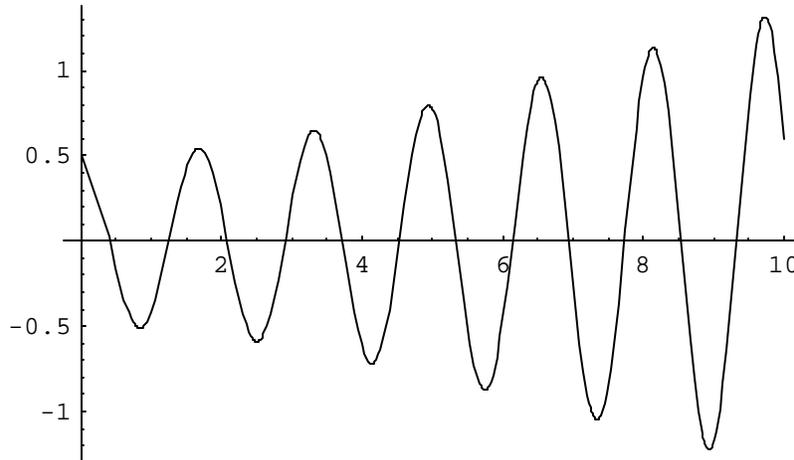


Figure 6.10. Trajectory of  $\theta(t)$  with zero damping but with frequency of the forcing function  $T_m$  equal to the natural frequency  $\omega$ .

The trajectory of  $\theta(t)$  in Fig. 6.10 is a case of resonance. According to Eisen (1988) such oscillations are not normally observed when walking or in other limb movements because of damping and neuromuscular feedback. However, he indicated that resonance occurs frequently in mechanical systems and has been responsible for many catastrophes. Eisen indicated that currents of air have served as driving forces setting off destructive oscillations in airplanes, steel factory chimneys, and other structures. He enumerated several examples of destructive resonance: the collapse of the Tacoma Narrows Bridge at Puget Sound in Washington in 1940 and the collapse of the Broughton suspension bridge in England in 1881. The disaster in England occurred when a column of soldiers marched in time over the bridge, thereby setting up a periodic frequency of rather large amplitude, inducing very large oscillations.

## 6.5 Stability test by the isocline and phase-plane methods

The second-order differential equation, such as (6.10), can be re-written into a system of first-order differential equations by introducing the variables  $x_1 = \mathbf{q}$  and  $x_2 = \mathbf{q}'$ .

With this transformation, (6.10) becomes

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\omega^2 x_1 - r x_2 \end{aligned} \tag{6.13}$$

The equilibrium state is a fixed point of the motion where  $x' = 0$ . The equilibrium state of (6.13) has the solution:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0; \quad \text{or} \quad \bar{x}_2 = \frac{v^2}{r} x_1, \quad \text{for any value of } x_1.$$

At equilibrium, the velocity and acceleration of the swinging body are at rest, that is  $\mathbf{q}' = 0$ ,  $\mathbf{q}'' = 0$ .

The problem of stability is a concern of any equilibrium state. Stability is related to the preservation of stationary-state situations in biological systems. The central notion of stability is this: If a system that is initially at rest is disturbed, does it return to rest in the long run or does it wander away? In other words, if a system is given a new initial value, will the orbit move towards the equilibrium state as time approaches infinity or will it at least remain in the vicinity of the equilibrium state? If it does, then the equilibrium state is stable, otherwise, it is unstable.

Isocline analysis

Isoclines are curves in the plane obtained by taking  $x' = 0$ . Along the isoclines, the rate of change of  $x$  is zero. The intersection of the isoclines is the equilibrium point. The isocline curves shown in Fig. 6.11 are plotted from

$$\bar{x}_2 = \frac{\omega^2}{r} x_1, \quad \text{for some values of } x_1. \quad \text{The isocline curves intersect at } \bar{x}_1 = 0, \quad \bar{x}_2 = 0$$

and therefore represent the equilibrium point of the system. This means that the oscillation will eventually stop.

```
r = 1;
w = 4;
x2 = - w^2/r*x1;
Figure10 = Plot[x2, {x1,-10,10}, PlotRange->{{-2,2},{-20,20}}, AxesLabel->{"x1", "x2"}]
```

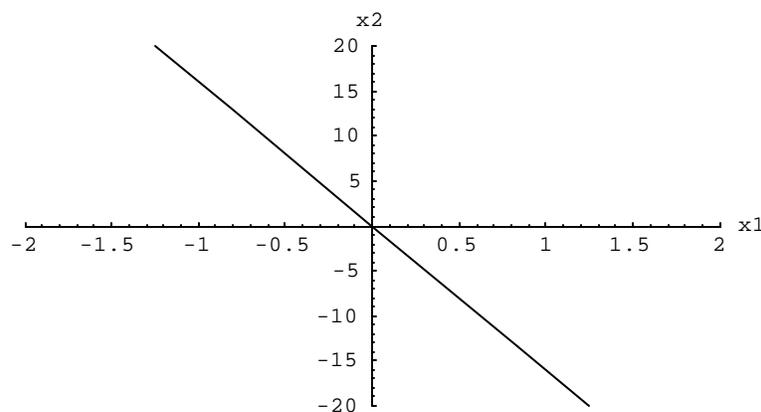


Figure 6.11. Isocline curve of  $x_1' = x_2$ ,  $x_2' = -v^2 x_1 - r x_2$  when  $r > 0$ .

Phase-plane analysis

Using the ParametricPlot function of Mathematica, we can see that the direction of orbit from the initial condition spirals down to the zero equilibrium point, as shown in the phase portrait of  $\bar{x}_1$  and  $\bar{x}_2$  in Fig. 6.12.

```
r = 1;
w=4;
eq1 = x1'[t] == x2[t];
eq2 = x2'[t] == -w^2*x1[t] - r*x2[t];
eq3 = NDSolve[{eq1, eq2, x1[0]==2, x2[0]==2}, {x1[t], x2[t]},
{t,0,10}];
Figure11 = ParametricPlot[ {x1[t], x2[t]} /. eq3, {t,0,10},
Compiled-> False, PlotRange->{{-3,3},{-10,10}}, AxesLabel->
{"x1", "x2"}]
```

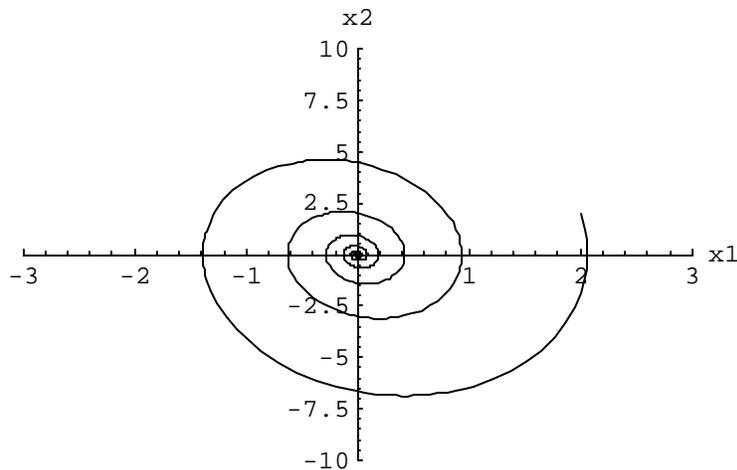


Figure 6.12. Phase-plane diagram of  $x_1' = x_2$ ,  $x_2' = -\nu^2 x_1 - r x_2$  when  $r > 0$ .

Therefore the zero equilibrium point of the system is stable. When  $r < 0$ , the zero equilibrium point, the only intersection of the isocline (Fig. 6.13) becomes unstable. The phase portrait of the system illustrated in Fig. 6.14 shows that the direction of the orbit is away from the zero equilibrium point.

```
r = -0.5;
x2 = - w^2/r*x1;
Figure12 = Plot[x2, {x1,-10,10}, PlotRange->{{-2,2},{-20,20}}, AxesLabel->{"x1", "x2"}]
```

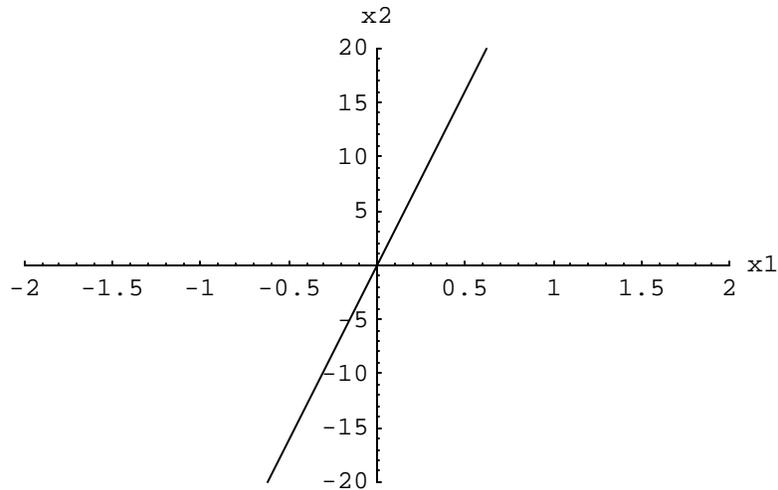


Figure 6.13. Isocline curve of  $x_1' = x_2$ ,  $x_2' = -v^2 x_1 - r x_2$  when  $r < 0$ .

```

r = -0.5;
w = 4;
eq1 = x1'[t] == x2[t];
eq2 = x2'[t] == -w^2*x1[t] - r*x2[t];
eq4 = NDSolve[{eq1, eq2, x1[0]==2, x2[0]==2}, {x1, x2},
{t,0,10}];
Figure13 = ParametricPlot[ {x1[t], x2[t]} /. eq4, {t,0,10},
Compiled-> False, PlotRange->{{-10,10},{-20,20}} 1

```

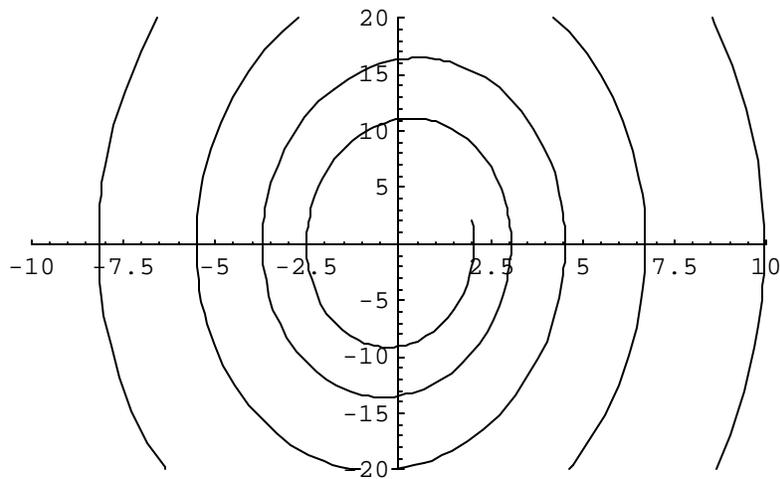


Figure 6.14. Phase-plane diagram of  $x_1' = x_2$ ,  $x_2' = -v^2 x_1 - r x_2$  when  $r < 0$ .

A phase-plane combination when  $r > 0$  and  $r < 0$  is illustrated in Fig. 6.15. The graph shows the region where the trajectory moves toward or away from the equilibrium point.

```
Show[Figure11, Figure13]
```

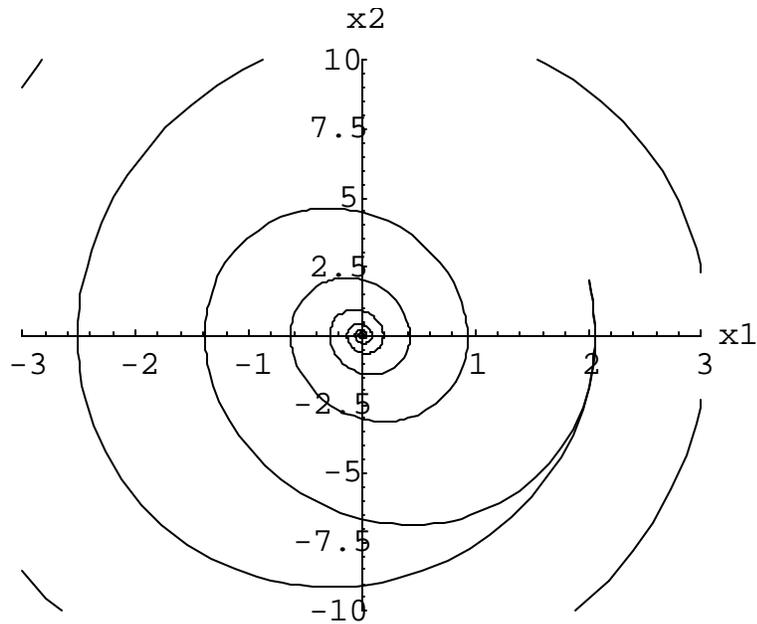


Figure 6.15. Phase-plane curves when  $r > 0$  and when  $r < 0$ .

When  $r$  is zero, the orbit goes into a limit cycle, as shown in Fig. 6.16. Limit cycles occur when the system contains a single unstable equilibrium state that is repelling in the sense that all orbits in the neighborhood of the equilibrium state move away from it (Beltrami, 1987).

```
r = 0;  
eq1 = x1'[t] == x2[t];  
eq2 = x2'[t] == -w^2*x1[t] - r*x2[t];  
eq4 = NDSolve[{eq1, eq2, x1[0]==2, x2[0]==2}, {x1, x2},  
{t,0,10}];  
Figure14 = ParametricPlot[ {x1[t], x2[t]} /. eq4, {t,0,10},  
Compiled-> False, PlotRange->{{-3,3},{-10,10}}]
```

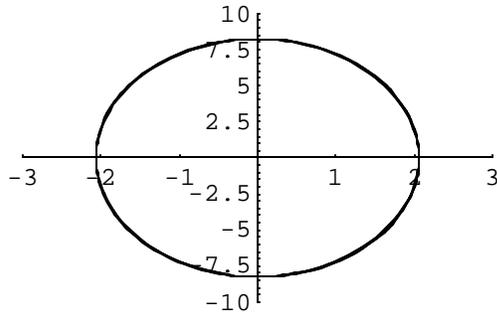


Figure 6.16. Phase-plane diagram of  $x_1' = x_2$ ,  $x_2' = -v^2 x_1 - r x_2$  when  $r=0$ . A limit cycle exists.

A combination of Figures 6.12, 6.14, and 6.16 is shown in Fig. 6.17 to demonstrate the trajectories of  $\theta(t)$  when  $r>0$  (stable equilibrium),  $r<0$  (unstable equilibrium), and  $r=0$  (existence of the limit cycle).

Show[Figure11, Figure13, Figure14]

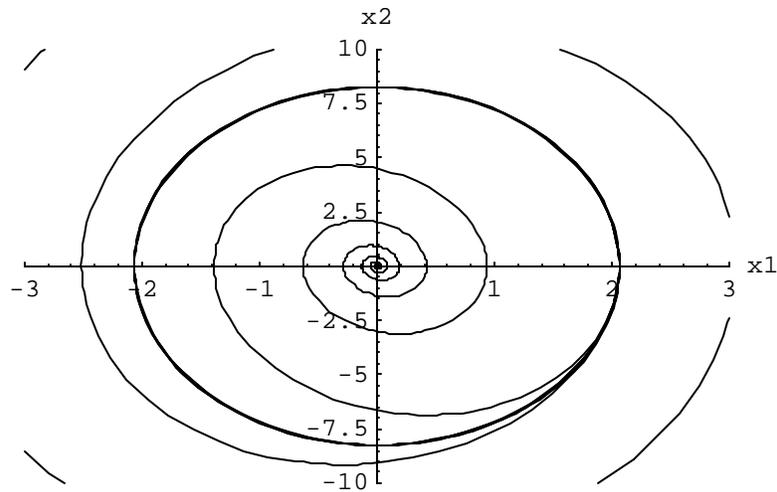


Figure 6.17. Trajectories of  $\theta(t)$  when  $r>0$  (stable equilibrium),  $r<0$  (unstable equilibrium), and  $r=0$  (existence of the limit cycle).

Example 6.1. Stability Test of the Predator-Prey System

The predator-prey system (equation 4.7) is reviewed as follows:

$$\frac{dx}{dt} = ax - bxy,$$

$$\frac{dy}{dt} = -cy + fxy,$$

where  $x$  and  $y$  represent prey and predator populations respectively; and the variables  $a$ ,  $b$ ,  $c$ , and  $f$  may represent biomass or population densities of the species. The steady state solution of equation (4.7) is derived by setting the derivatives to zero, that is:

$$\frac{dx}{dt} = 0; \quad \frac{dy}{dt} = 0$$

which leads to

$$x = \frac{c}{f}; \quad y = \frac{a}{b} \tag{6.14}$$

We can see that the equilibrium population of the prey depends directly on the death rate of the predator and inversely on the efficiency of predation. The higher the efficiency of predation, the lower is the prey population. On the other hand, the predator population depends directly on the growth rate of the prey.

Consider now two species where one is the food source of the other (Beltrami, 1987). The prey has an unlimited food source while the predator feeds on the prey. Let the prey population be represented as  $P_1$  and the predator population be  $P_2$ . In this situation,  $P_1$  is the only food source of  $P_2$ , so that when  $P_1$  is abundant,  $P_2$  is able to increase in its population, but in the absence of  $P_1$ ,  $P_2$  dies out at a constant per-capita rate of  $c > 0$ . When predation is absent,  $P_1$  is assumed to grow logistically. In the presence of mutual interaction,  $P_1$  population decreases while  $P_2$  population increases. The interaction between the two species is assumed to occur in proportion to the total number of possible ways in which  $P_1$  and  $P_2$  meet. This concept is mathematically modeled as the product of the two populations, that is,  $P_1P_2$ . The predator-prey model assumes no outside interference and is given as follows:

$$\begin{aligned} \frac{dP_1}{dt} &= rP_1 \left( 1 - \frac{P_1}{L} \right) - aP_1P_2 \\ \frac{dP_2}{dt} &= -cP_2 + bP_1P_2 \end{aligned} \tag{6.15}$$

Let the *Microtus* voles be the prey (and its population  $P_1$ ) and the least weasel (mustelids) be the predator (and its population  $P_2$ ). Let the following parameters be defined: the annual growth rate of  $P_1$  is 5.4 voles  $\text{yr}^{-1}$  ( $r$ ); the carrying capacity of  $P_1$  is 100  $\text{ha}^{-1}$  ( $L$ ); the variable  $a$  is the consumption factor of the predator ( $P_2$ ) on the prey ( $P_1$ ), where  $a=4$ ; the variable  $b$  is the per-capita growth rate of  $P_2$  resulting from its catch of  $P_1$ , where  $b=10\%$  of  $a$ ; and  $c$  is the per-capita death rate of  $P_2$ . The variable  $c$  can also mean to be the harvesting rate of  $P_2$ . Estimate the population over 10 years when the initial populations are 4 for  $P_1$  and 2 for  $P_2$ .

```

r=5.4; L=100; a=4; c = 2.5; b = 0.1*a;
prey = P1'[t] == r*P1[t] (1 - P1[t]/L) - a*P1[t]*P2[t];
pred = P2'[t] == - c*P2[t] + b*P1[t]*P2[t];
eq1 =NDSolve[{prey,pred, P1[0]==4, P2[0]==2}, {P1[t], P2[t]},
{t,0,10}];
p1=Plot [ Evaluate [{P1[t],P2[t]} /. eq1], {t,0,10},
AxesLabel -> {"Years", "Voles and weasel populations"} ]

```

Voles and weasel populations

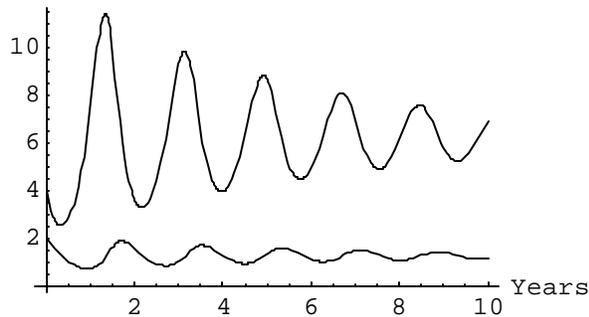


Figure 6.18. Growth curves of the voles and weasels.

#### Steady State Analysis

What is the population of each species at equilibrium? We can answer this question by setting  $\frac{dP_1}{dt}$  and  $\frac{dP_2}{dt}$  equal to zero.

$$\frac{dP_1}{dt} = rP_1 \left( 1 - \frac{P_1}{L} \right) - aP_1P_2 = 0$$

$$\frac{dP_2}{dt} = -cP_2 + bP_1P_2 = 0$$

The solutions are:  $P_1 = \frac{c}{b}$ ;  $P_2 = \frac{r}{a} \left( 1 - \frac{c}{bL} \right)$ ;

The population of  $P_1$  is directly proportional to the death rate of  $P_2$  and inversely proportional to the growth rate of  $P_2$ . The population of  $P_2$  is directly proportional to the growth rate of  $P_1$ . Both populations are directly affected by the per-capita death rate of  $P_2$ , which could also mean the harvesting rate of  $P_2$ .  $P_1$  will experience positive feedback when  $c$  increases and negative feedback when  $c$  decreases.

### Isocline analysis

```
PP1 = c/b;  
PP2 = r/a*(1 - c/(b*L));  
points={{PP1,0}, {PP1,PP2}, {0,PP2}};  
p2=ListPlot[points, PlotJoined->True, AxesLabel->{"Voles",  
"Weasels"}]
```

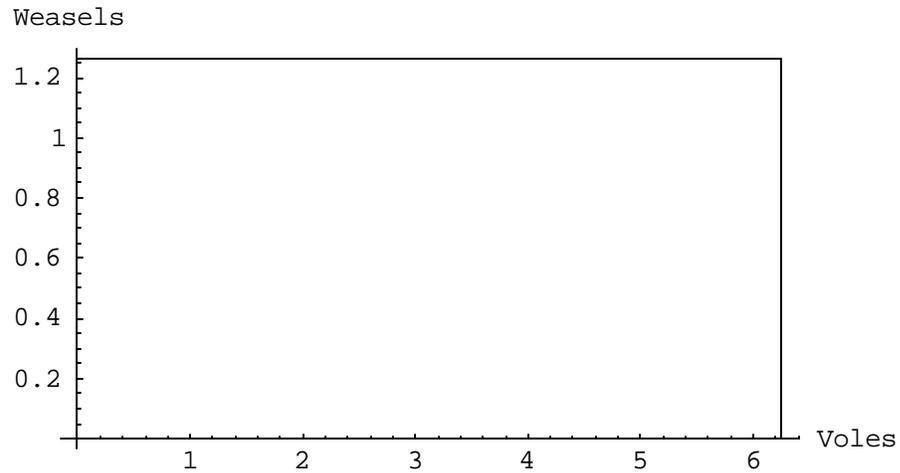


Figure 6.19. Isocline curves of  $\frac{dP_1}{dt} = 0$  and  $\frac{dP_2}{dt} = 0$ .

### Phase-plane analysis

```
p3=ParametricPlot[ {P1[t], P2[t]} /. eq1, {t,0,10},  
PlotRange->{{0,12},{0,2}},  
Compiled -> False, AxesLabel -> {"Voles", "Weasels"}]
```

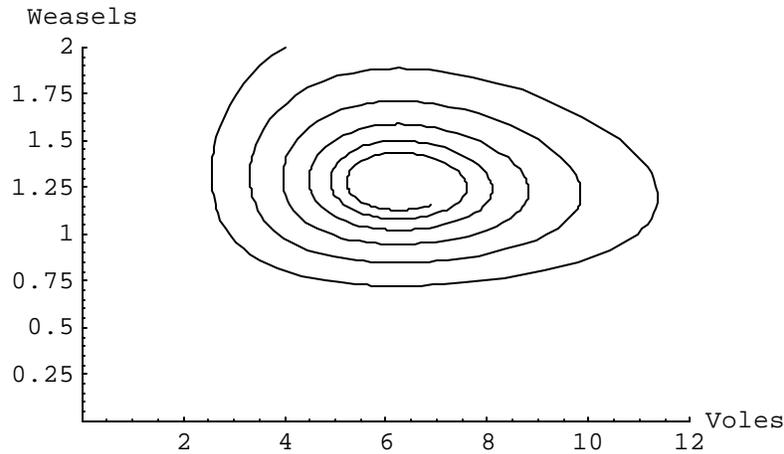


Figure 6.20. Phase-plane curve of  $\frac{dP_1}{dt}$  and  $\frac{dP_2}{dt}$ .

Show[p2,p3]

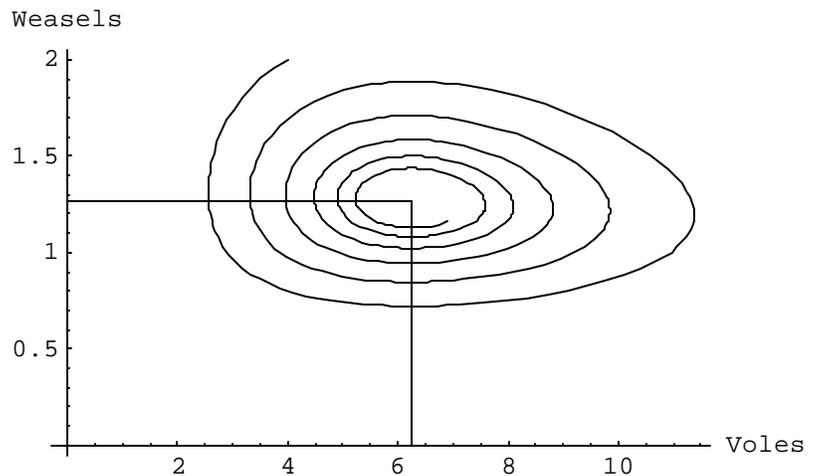


Figure 6.21. Isocline and phase-plane curves of  $\frac{dP_1}{dt}$  and  $\frac{dP_2}{dt}$ .

Figure 6.21 shows that the equilibrium point is stable.

### Exercises

1. *Lotka-Volterra's predator-prey system.* Consider two species existing in the same environment having a host-parasite relationship. In the absence of the parasite (P), the rate of growth of the host population (H) is proportional to its population. In the same manner, if there are no hosts, the parasites die due to lack of food and the rate of population reduction is proportional to its population. If hosts and parasites are interacting, the mortality of the host and the population growth of the parasite are proportional to the encounter between

the host and the parasite. The encounter can be modeled as proportional to the product of their respective population sizes.

- a. Formulate the differential equations.
  - b. Show graphically the oscillation in the two populations if the following parameters are defined: growth rate of the host is 0.8; death rate of the parasite is 0.8; decline factor on host population due to “predation” by parasite is 0.05; and increase factor on parasite population due to predation is 0.04. Let the initial population of the host be 20 and the initial population of the parasite be 10.
  - c. Analyze the stability of the system by the isocline and phase-plane methods. Is the system stable or not?
  - d. What are the steady state populations?
2. *Water flow problem* (adapted from Beltrami, 1988, p. 12). Consider a volume of water flowing into a tank at a constant rate  $f$  ( $\text{m}^3 \text{t}^{-1}$ ). Water evaporates from the tank at a rate which is proportional to  $v^{2/3}$ , where  $v$  is the volume of the tank and the proportionality constant is  $k > 0$ . Apply the conservation of mass principle to formulate the model.
- a. Formulate the differential equations.
  - b. Find the equilibrium state.
  - c. Check for stability of the system by the isocline and phase-plane methods.
3. *Water flow, Part II*. Water enters a cylindrical tank at a constant rate  $f$ , as above (Beltrami, 1988, p. 14). The tank has a hole in the bottom and so water flows out through it. Let  $h$  be the depth of the water in the tank. It is known that the rate at which water flows through the hole (distance per unit time) is proportional to  $h^{1/2}$ . Suppose the tank has a constant cross-sectional area  $A$  and that the hole has an area  $A_h$ .
- a. Find the differential equation for  $h$ .
  - b. Solve for the equilibrium state.
  - c. Check for stability of the system by the isocline and phase-plane methods.
4. *Water flow, Part III*. Reformulate exercise 6.2 by letting the rate at which water enters the tank be proportional to the depth of water  $h$  (Beltrami, 1988, p. 59). Show that the nontrivial equilibrium state is unstable.
5. *Lake pollution* (adapted from Beltrami, 1988, p. 13). The situation is about an organic pollutant ( $x_1$ ) entering a lake at a constant rate,  $s$ . Bacterial action metabolizes (decomposes) the pollutant at a rate proportional to its mass, with

$k_1 > 0$  as the proportionality constant. The dissolved oxygen in the lake waters ( $x_2$ ) is used up at the same rate that the pollutant decomposes. However, re-aeration (oxygen from the atmosphere entering the lake through the surface-to-air contact) at a rate proportional to the difference between the maximum dissolved oxygen level the lake can support ( $x_m$ ) and its current actual value. The re-aeration proportionality constant is  $k_2 > 0$ .

- a. Write the differential equations for  $x_1$  and  $x_2$ .
  - b. Solve for the equilibrium state.
  - c. Check for system stability.
6. Do an isocline analysis of the SIRS model presented in equations (4.16) of Chapter 4.

## References

- Beltrami, E. 1987. *Mathematics for Dynamic Modeling*. Academic Press, Boston. 277 pp.
- Eisen, M. 1988. *Mathematical Methods and Models in the Biological Sciences. Linear and One-Dimensional Theory*. Prentice Hall, New Jersey. 299 pp
- Minnesota Dept. of Natural Resources, 1993. Status of wildlife populations, Fall 1993 and 1981-1992 hunting and trapping harvest statistics. M.H. Dexter, compiler. Unpublished report.
- Weidner, R.T. and R.L. Sells. 1965. *Elementary Classical Physics, Vol. 1*. Allyn and Bacon, Inc., Boston).



## Sustainability

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### PRINCIPLE

- Sustainability requires responsible control and management.
- 

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### TOPICS

- Sustainable harvesting of one species
  - Fisheries management
  - Sustainable harvesting of two interacting populations
  - Nutrient loading
- 

In Chapter 1, we were introduced to the concept of sustainability as a philosophical paradigm that addresses economic growth and development within the limits set by ecology. Any population control decision we make today will have consequences in the future, good or bad. Chapter 4 introduced us to the logistic function to model the dynamics of a population. We learned that if the initial population is below the carrying capacity, the rate of change of the population ( $\frac{dP}{dt}$ ) will be positive and the population will increase to the carrying capacity (L); if the initial population exceeds L, then  $\frac{dP}{dt}$  will be negative and the population will decline to the value of L. The logistic equation is as follows:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{L}\right) \quad (7.1)$$

The major reason for understanding the behavior of biological dynamic systems is to be able to control and manage the system for sustainability so that the future generation can have access to the same opportunities as we have in the present generation. Management and control may be expressed in terms of harvesting or stocking.

### 7.1 Sustainable Harvesting of One Species

Suppose harvesting of fish, wildlife animals, and other species is allowed. In this case, the population has negative input. In this situation, there are two possible management strategies that can be adopted. One is to have a fixed harvest rate, that is, a harvest rate at a constant value. This strategy provides the hunters a quota of kills each year. This situation does not consider the existing population size. The second management strategy involves harvesting at a fixed effort, with the harvest rate linearly proportional to the population size. This management strategy is environmentally friendly as it is sensitive to the sustainability of the species population.

Consider now applying the second management strategy to the pheasant population where the harvesting rate is proportional to the population size. It is not desirable to harvest less than the optimum because it may mean reduced economic return (for hunting fees and hunting-related income). On the other hand, we don't want to harvest more than the optimum because it may mean potential extinction of the animal species. We will use a nonhomogeneous logistic equation where the forcing function is in proportion to the population size of the species. The equation can be written as follows:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{L}\right) - EP \quad (7.2)$$

where E is the fixed effort to be spent by the hunter and EP is the allowable harvest rate. The optimum harvest rate is being desired.

First, we determine the population size where the maximum rate of change of population occurs before harvesting is introduced. To do this, we take the derivative of the logistic function with respect to the population size, set it to zero, and solve for P, that is,

$$f(P) = rP\left(1 - \frac{P}{L}\right) \quad (7.3)$$

$$\frac{df(P)}{dP} = r - \frac{2r}{L}P \quad (7.4)$$

$\frac{df(P)}{dP} = 0$  leads to

$$P_m = \frac{L}{2} \quad (7.5)$$

where  $P_m$  is the maximum population size. We can check whether our solution is the maximum by taking the second derivative of equation (7.4). Since the second derivative of equation (7.4) is negative-valued, then equation (7.5) is indeed the maximum population size, and not the minimum.

Graphically, we can show that  $f(P) = \frac{dP}{dt}$  is maximum when  $P = \frac{L}{2}$ , as shown in Fig. 7.1.

```

r = 0.5;
L = 3000;
fP = r*P*(1-P/L);
Plot [fP, {P,0,3000},PlotRange->{{0,3000},{0,400}}, AxesLabel->{"Population", "f(P)"}]

```

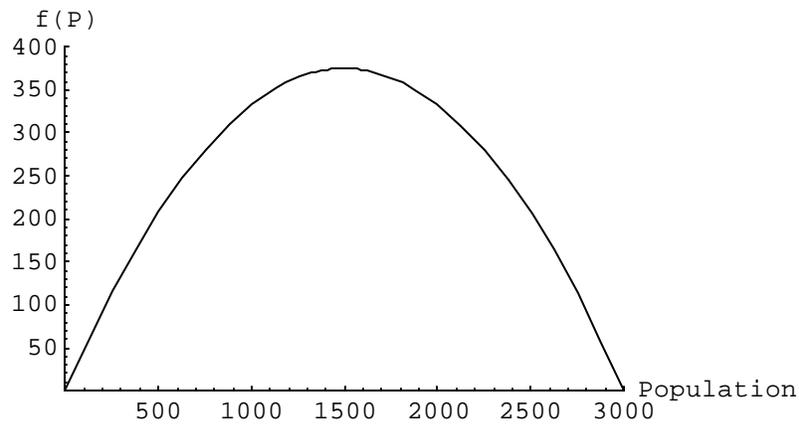


Figure 7.1. Rate of change of the population as a function of the population size.

To find the optimum harvest rate,  $EP$ , at steady state, we take the function

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{L}\right) - EP$$

and solve for  $E$  and  $P$  at steady state. This leads to

$$E = r(1 - P/L) \quad (7.6)$$

The optimum effort would be achieved when  $\frac{dP}{dt}$  is maximum, that is when  $P_m = L/2$ . Therefore,

$$E_m = r(1 - P_m/L) = r/2 \quad (7.7)$$

And the optimum harvesting rate is

$$E_m P_m = rL/4 \quad (7.8)$$

The dynamics of a population with optimum harvesting is demonstrated in the following Mathematica exercises.

Maximum harvest rate; maximum initial population

```
r=0.5;
L=3000;
u=r*L/4;
eq1=NDSolve[{P'[t]==r P[t] (1 - P[t]/L) - u, P[0]==L/2},P[t],{t,0,200}];
p1=Plot [Evaluate [P[t] /. eq1], {t,0,200}, PlotRange->{0,2000},AxesLabel->{ "time",
"Population"}]
```

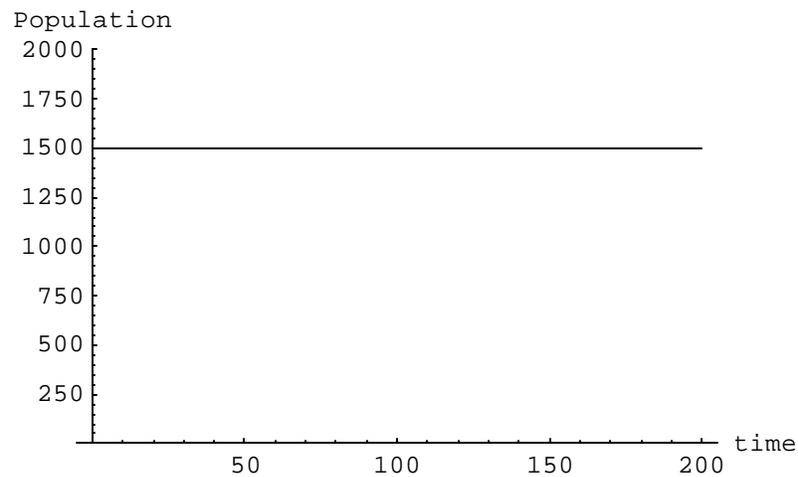


Figure 7.2. Maximum harvest rate; maximum initial population. The system is sustainable.

Harvest rate one more than maximum; maximum initial population  
One unit more than optimum harvest; initial population is equal to  $L/2$

```
r=0.5;
L=3000;
u=r*L/4 + 1;
eq1=NDSolve[{P'[t]==r P[t] (1 - P[t]/L) - u, P[0]==L/2},P[t],{t,0,120}];
p2=Plot [Evaluate [P[t] /. eq1], {t,0,120}, PlotRange->{0,2000},AxesLabel->{ "time",
"Population"}]
```

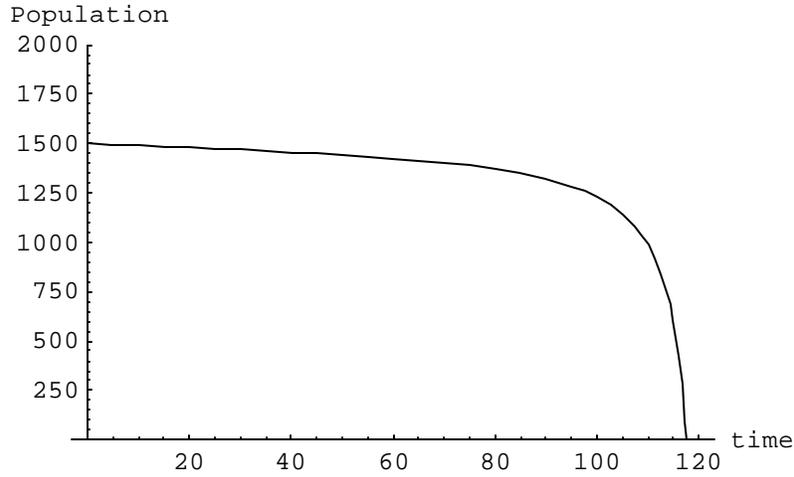


Figure 7.3. Harvest rate more than optimum; maximum initial population. The system is not sustainable.

Harvest rate one less than maximum; maximum initial population  
 One unit more than optimum harvest; initial population is equal to  $L/2$

```

r=0.5;
L=3000;
u=r*L/4 - 1;
eq1=NDSolve[{P'[t]==r P[t] (1 - P[t]/L) - u, P[0]==L/2},P[t],{t,0,120}];
p2=Plot [Evaluate [P[t] /. eq1], {t,0,120}, PlotRange->{0,2000},AxesLabel->{ "time",
"Population"}]

```

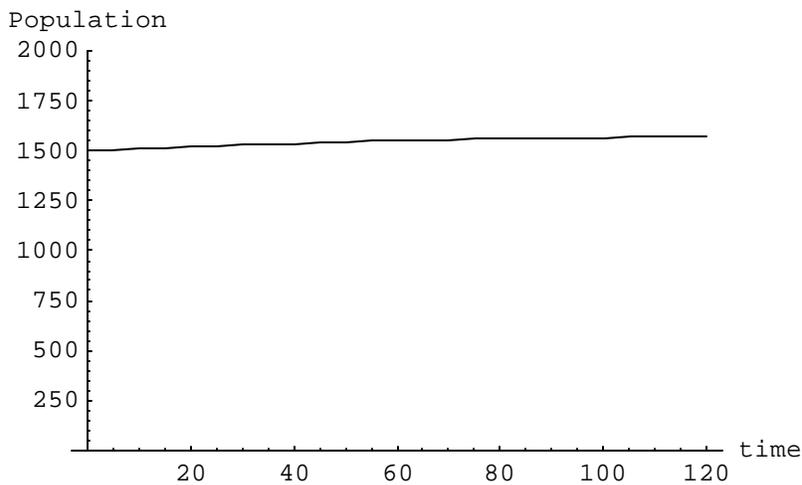


Figure 7.4. Harvest rate less than optimum; maximum initial population. The system is not efficient since more harvesting could have been done.

## 7.2 Sustainable Harvesting Proportional to Effort

Fisheries are managed for the benefit of humankind. The fundamental objective of fisheries management is to ensure a sustainable production of fish. Objectives of fisheries management can be classified as biological, economic, recreational, and social. In this exercise, we shall be modeling the dynamics of fish stock as a function of habitat condition, fish species, harvesting effort, economic condition surrounding fish market, and technology of fish harvesting. Factors that affect fish stock can be classified as environmental, economic, and technological. Environmental factors may include pollution (oil spills, pesticides, fertilizers, acid rain, rising salinity and acidity), and drying up. Economic factors may include consumer's demand, cost of entry into the fishing industry (taxes, cost of fuel, fishing license), and international trade agreements. Technological factors may include type of technology (electronic equipment, type of fishing boats, geopositioning technologies), and fish tagging.

For our study, the following equations are used:

$$\begin{aligned} \frac{dP}{dt} &= rP \left( 1 - \frac{P}{L} \right) - vEP \\ \frac{DE}{dt} &= aE(pvP - c) \end{aligned} \quad (7.9)$$

where  $P$  is the fish population (thousands of fish),  $E$  is the harvesting effort (net throws  $\text{yr}^{-1}$ ),  $r$  is the specific growth rate (per thousand fish  $\text{yr}^{-1}$ ),  $L$  is the carrying capacity (thousands of fish),  $v$  is the “catchability coefficient” (success of catch/net throw),  $a < 0$  is some constant of proportionality,  $p$  is the price of 1000 fish, and  $c$  is the cost per net throw. Model parameters will be decreased or increased by 50% and compared with default values. The dynamic behavior of the fish stock is observed through time. Phase-plane curves are also generated. The fish stock analyses are done as follows.

```

r=0.03; L=3000; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve [{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g1=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]
    
```

Fish Pop. & Effort

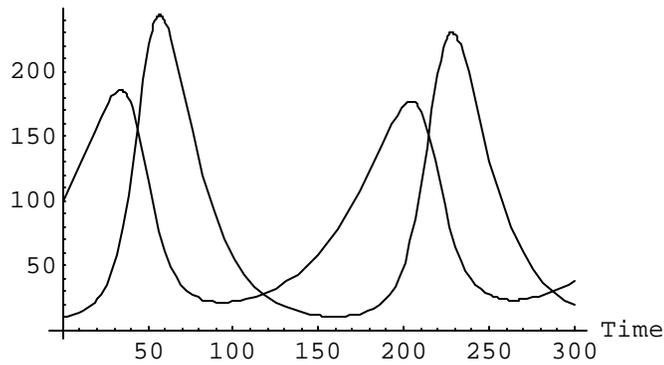


Figure 7.5. Interaction between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

Decrease price of fish by 50%

```

r=0.03; L=3000; v=0.0004; a=0.0002; p=5000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2=Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g2=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]
    
```

Fish Pop. & Effort

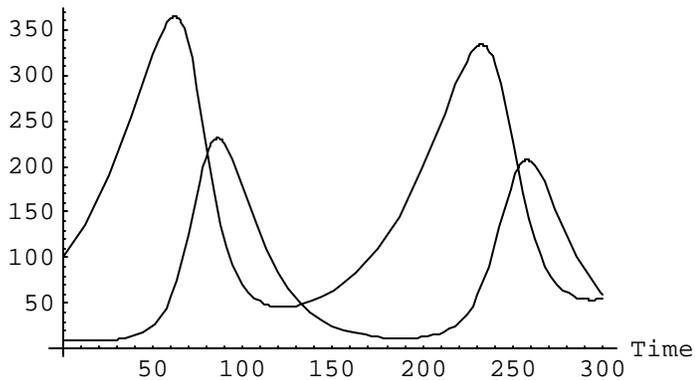


Figure 7.6. Interaction between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=5000$ , and  $c=300$ .

Increase price of fish by 50%

```

r=0.03; L=3000; v=0.0004; a=0.0002; p=15000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g3=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]

```

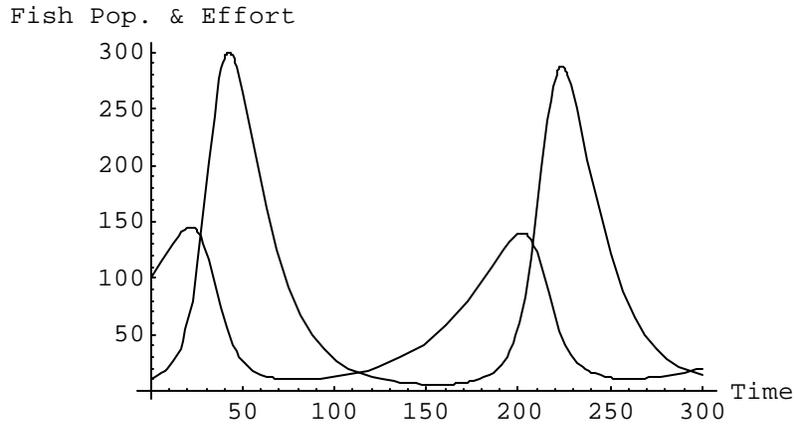


Figure 7.7. Interaction between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=15000$ , and  $c=300$ .

Phase-plane analysis

```

r=0.03; L=3000; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g4=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish Pop.,";"Effort"}]

```

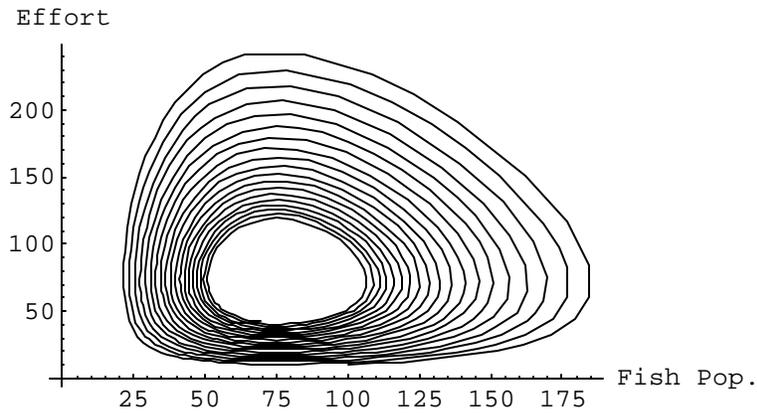


Figure 7.8. Phase-plane analysis between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

```

r=0.03; L=3000; v=0.0004; a=0.0002; p=5000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g5=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish
Pop.", "Effort"}]

```

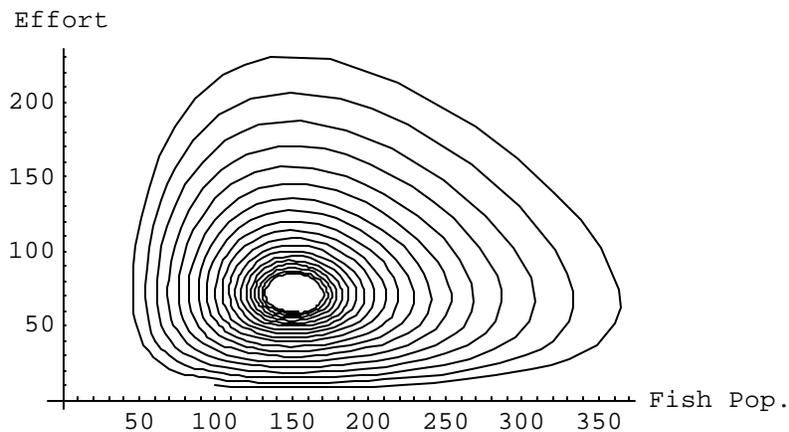


Figure 7.9. Phase-plane analysis between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=5000$ , and  $c=300$ .

```

r=0.03; L=3000; v=0.0004; a=0.0002; p=15000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g6=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish Pop.", "Effort"}]

```

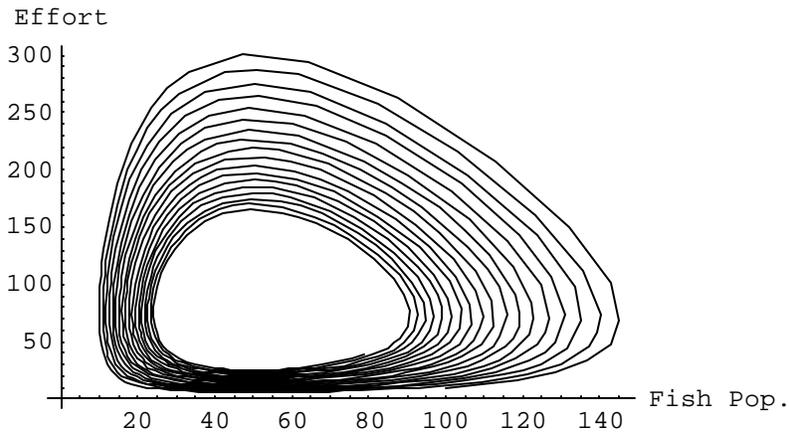


Figure 7.10. Phase-plane analysis between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=15000$ , and  $c=300$ .

```
Show[g4, g5, g6]
```

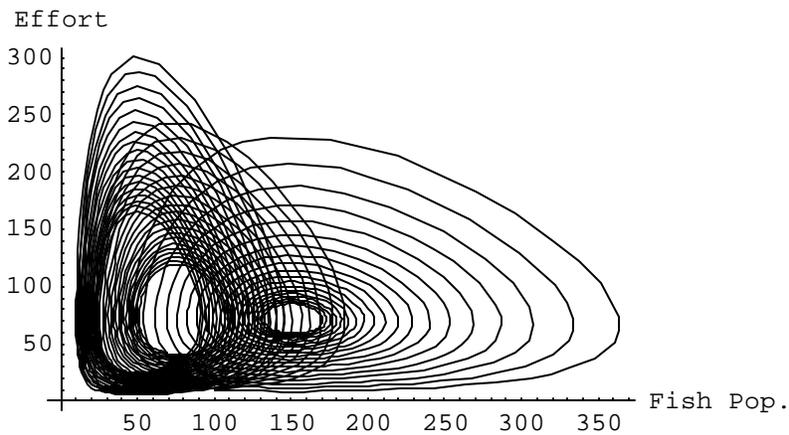


Figure 7.11. Phase-plane combination between fish population and effort when  $p=10000$ ,  $p=5000$ , and  $p=15000$ .

Decrease catchability coefficient by 50%.

```

r=0.03; L=3000; v=0.0002; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve [{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g7=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]

```

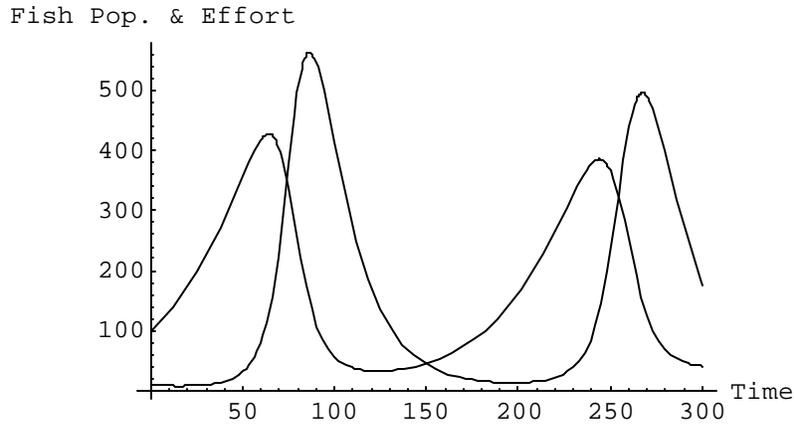


Figure 7.12. Interaction between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0002$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

Increase catchability coefficient by 50%

```

r=0.03; L=3000; v=0.0006; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve [{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g8=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]

```

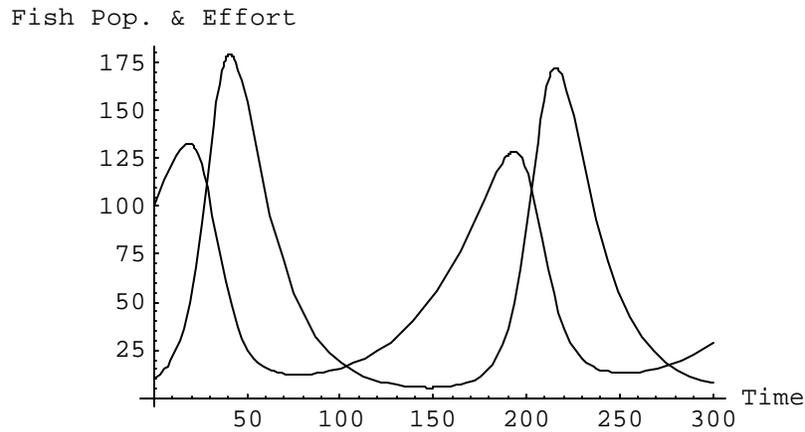


Figure 7.13. Interaction between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0006$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

```

r=0.03; L=3000; v=0.0002; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g9=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish
Pop.,"Effort"}]

```

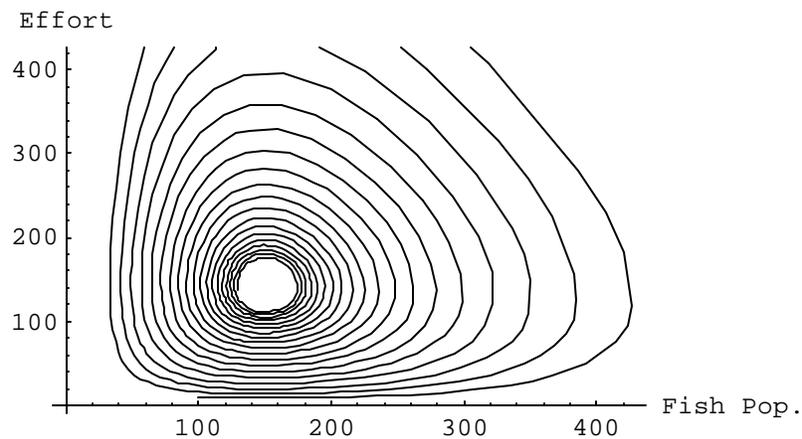


Figure 7.14. Phase-plane analysis between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0002$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

```

r=0.03; L=3000; v=0.0006; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g10=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish Pop.", "Effort"}]

```

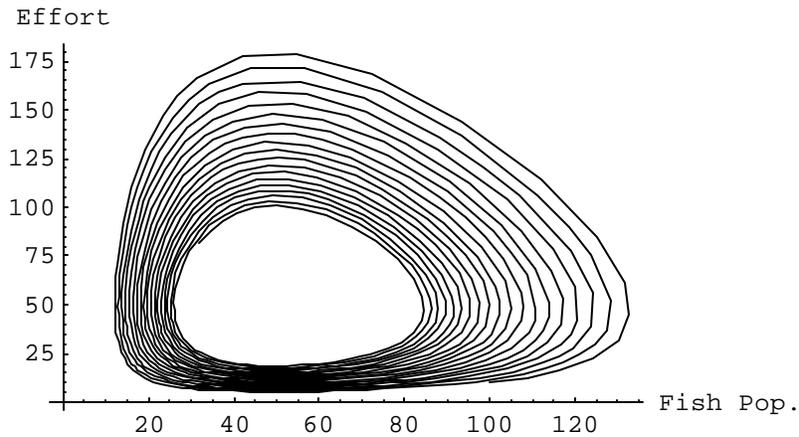


Figure 7.15. Phase-plane analysis between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0006$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

```
Show[g4, g9, g10]
```

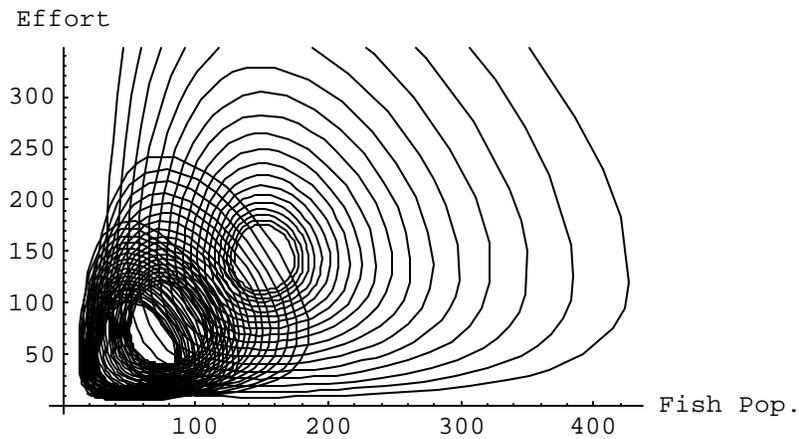


Figure 7.16. Phase-plane combination between fish population and effort when  $v=0.0002$ ,  $v=0.0004$ , and  $v=0.0006$ .

Decrease specific growth by 50%

```
r=0.015; L=3000; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g11=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]
```

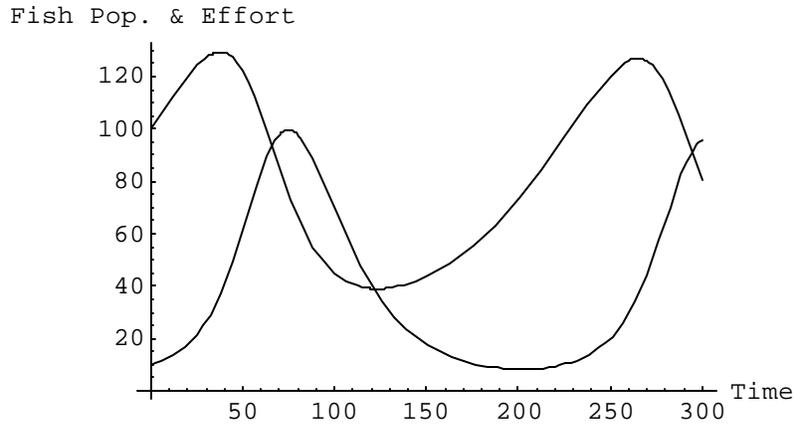


Figure 7.17. Interaction between fish population and effort when  $r=0.015$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

Increase specific growth by 50%

```
r=0.045; L=3000; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g12=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]
```

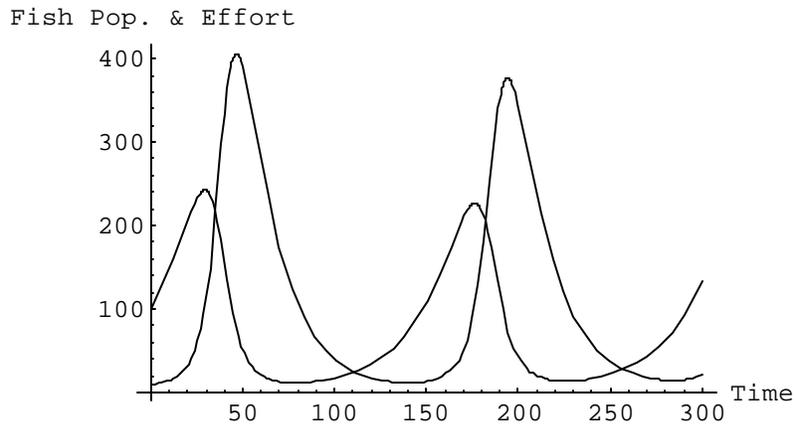


Figure 7.18. Interaction between fish population and effort when  $r=0.045$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

### Phase-plane analysis

```

r=0.015; L=3000; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve [{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g13=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish
Pop.,"Effort"}]

```

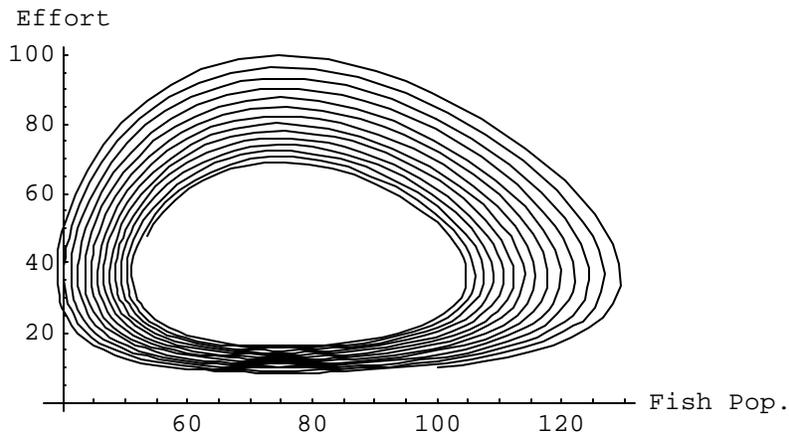


Figure 7.19. Phase-plane analysis between fish population and effort when  $r=0.015$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

```

r=0.045; L=3000; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g14=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish Pop.", "Effort"}]

```

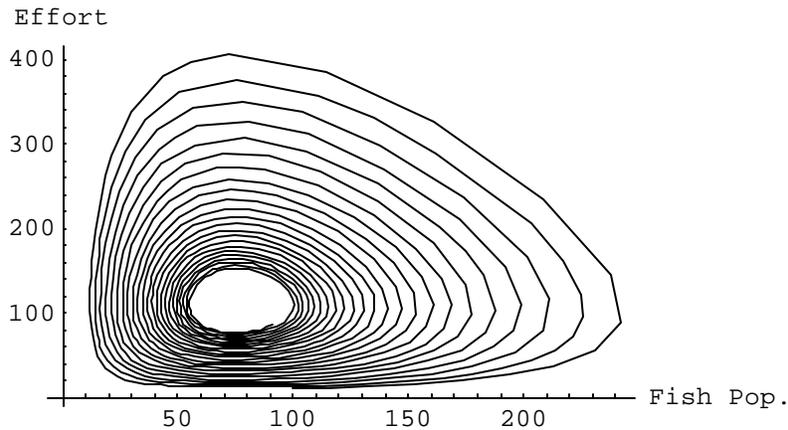


Figure 7.20. Phase-plane analysis between fish population and effort when  $r=0.045$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

```
Show[g4,g13,g14]
```

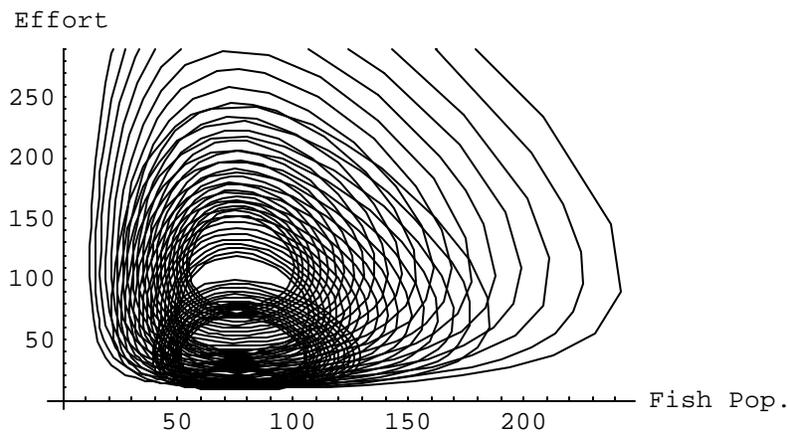


Figure 7.21. Phase-plane combination between fish population and effort when  $r=0.015$ ,  $r=0.03$ , and  $r=0.045$ .

Decrease cost per net throw by 50%

```

r=0.03; L=3000; v=0.0004; a=0.0002; p=10000; c=150;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g15=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]

```

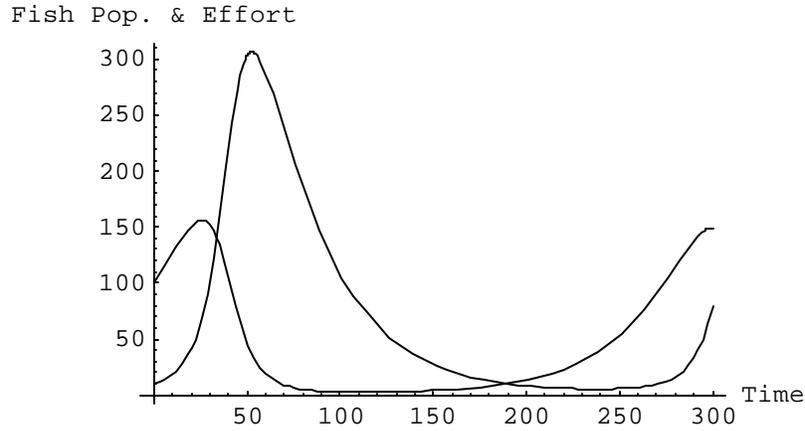


Figure 7.22. Interaction between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=150$ .

Increase cost per net throw by 50%

```

r=0.03; L=3000; v=0.0004; a=0.0002; p=10000; c=450;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g16=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]

```

Fish Pop. & Effort

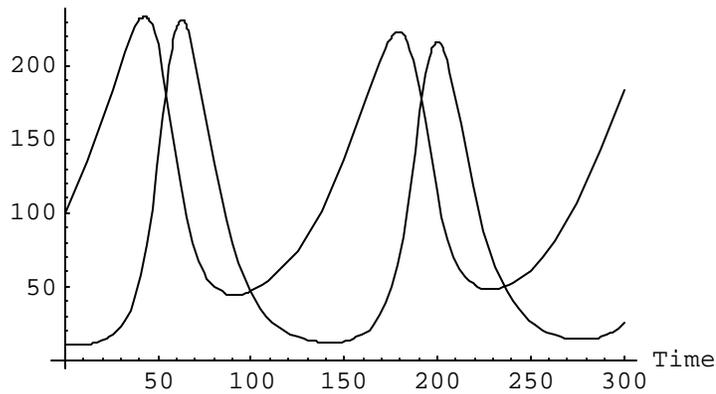


Figure 7.23. Interaction between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=450$ .

Phase-plane analysis

```
r=0.03; L=3000; v=0.0004; a=0.0002; p=10000; c=150;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve [{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g17=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish
Pop.,"Effort"}]
```

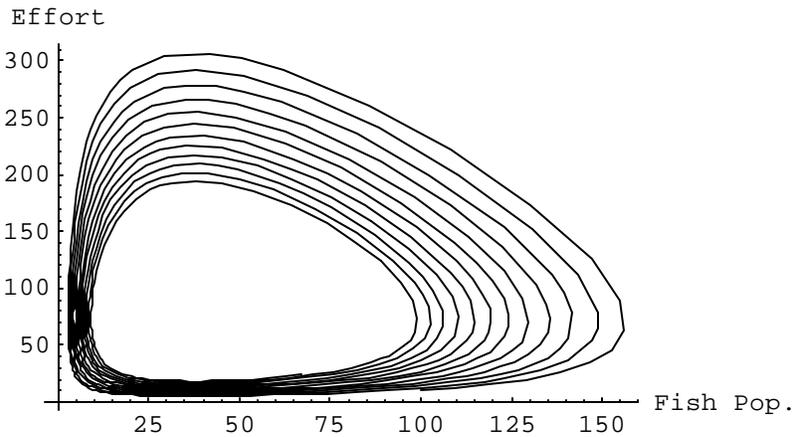


Figure 7.24. Phase-plane analysis between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=150$ .

```

r=0.03; L=3000; v=0.0004; a=0.0002; p=10000; c=450;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g18=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish
Pop.", "Effort"}]

```

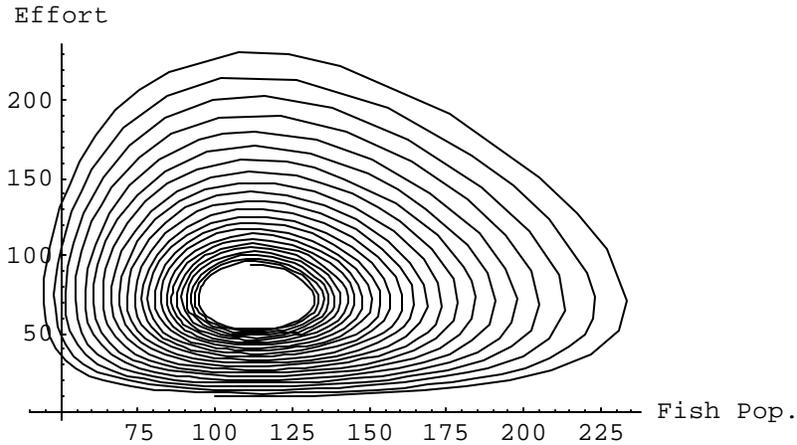


Figure 7.25. Phase-plane analysis between fish population and effort when  $r=0.03$ ,  $L=3000$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=450$ .

```
Show[g4,g17,g18]
```

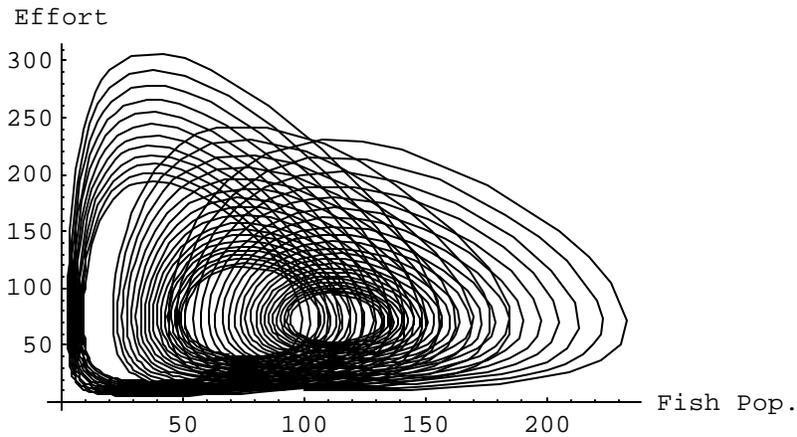


Figure 7.26. Phase-plane combination between fish population and effort when  $c=150$ ,  $c=450$ , and  $c=450$ .

Decrease carrying capacity by 50%

```

r=0.03; L=1500; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve [{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g19=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]

```

Fish Pop. & Effort

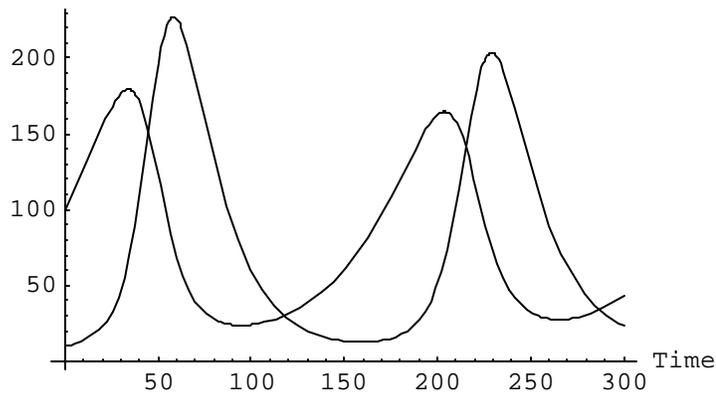


Figure 7.27. Interaction between fish population and effort when  $r=0.03$ ,  $L=1500$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

Increase carrying capacity by 50%

```

r=0.03; L=4500; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Ef[t] P[t];
eq2= Ef'[t]==a Ef[t] (p v P[t] - c);
eq3=NDSolve [{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,300}, MaxSteps->2000];
g20=Plot[Evaluate[{P[t], Ef[t]} /. eq3], {t,0,300}, AxesLabel->{"Time", "Fish Pop. & Effort"}]

```

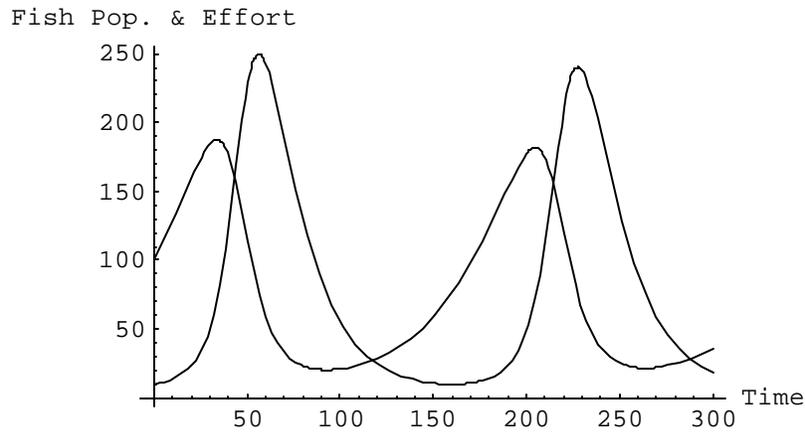


Figure 7.28. Interaction between fish population and effort when  $r=0.03$ ,  $L=4500$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

Phase-plane analysis

```

r=0.03; L=1500; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Eff[t] P[t];
eq2= Ef'[t]==a Eff[t] (p v P[t] - c);
eq3=NDSolve [{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g21=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish
Pop.,"Effort"}]

```

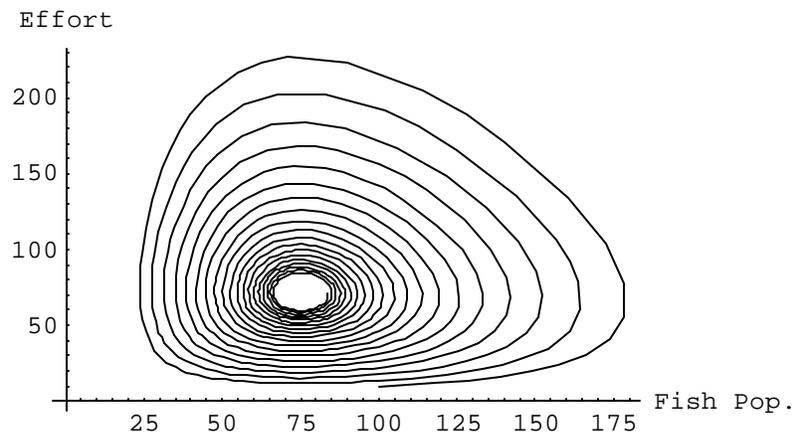


Figure 7.29. Phase-plane analysis between fish population and effort when  $r=0.03$ ,  $L=1500$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

```

r=0.03; L=4500; v=0.0004; a=0.0002; p=10000; c=300;
eq1=P'[t]==r P[t] (1-P[t]/L) - v Eff[t] P[t];
eq2= Ef'[t]==a Eff[t] (p v P[t] - c);
eq3=NDSolve [{eq1, eq2, P[0]==100, Ef[0]==10}, {P, Ef}, {t,0,3000}, MaxSteps->2000];
g22=ParametricPlot[{P[t], Ef[t]} /. eq3, {t,0,3000}, Compiled->False, AxesLabel->{"Fish
Pop.", "Effort"}]

```

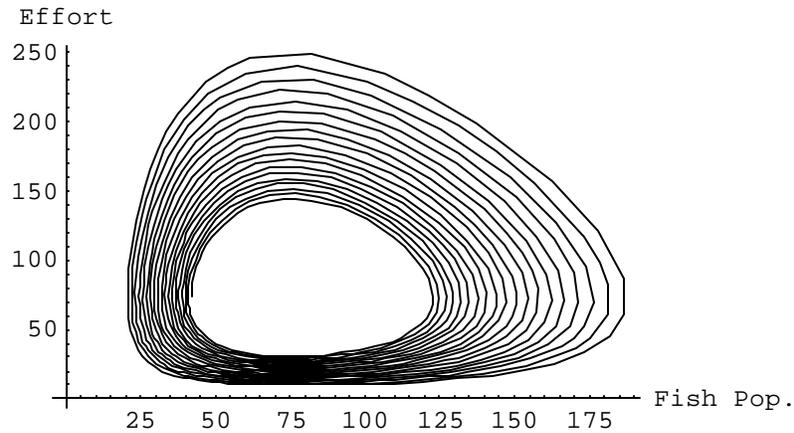


Figure 7.30. Phase-plane analysis between fish population and effort when  $r=0.03$ ,  $L=4500$ ,  $v=0.0004$ ,  $a=0.0002$ ,  $p=10000$ , and  $c=300$ .

Show[g4,g21,g22]

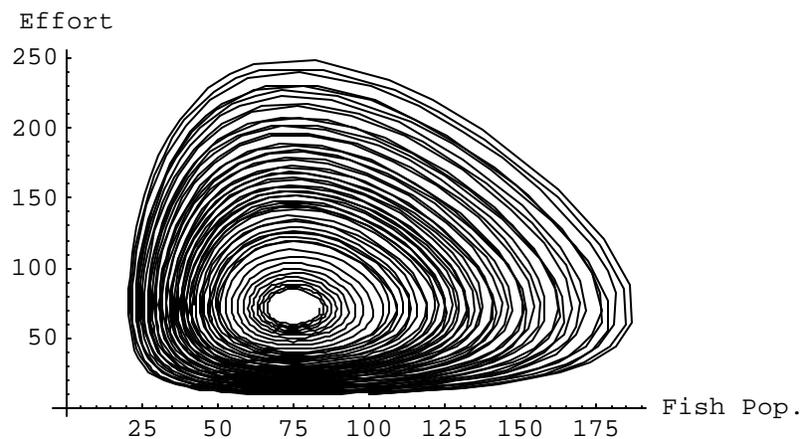


Figure 7.31. Phase-plane combination between fish population and effort when  $L=1500$ ,  $L=3000$ , and  $L=4500$ .

## 7.3 Sustainable Harvesting of Two Interacting Populations

Consider the predator-prey relationship of the *Microtus voles* (as prey) and the least weasel (as predator), in Chapter 4. The following parameters were defined: the annual growth rate of  $P_1$  is 5.4 voles  $\text{yr}^{-1}$  ( $r$ ); the carrying capacity of  $P_1$  is 100  $\text{ha}^{-1}$  ( $L$ ); the variable  $a$  is the consumption factor of the predator ( $P_2$ ) on the prey ( $P_1$ ), where  $a=4$ ; the variable  $b$  is the per-capita growth rate of  $P_2$  resulting from its catch of  $P_1$ , where  $b=10\%$  of  $a$ ; and  $c$  is the per-capita death rate of  $P_2$ . The variable  $c$  can also mean to be the harvesting rate of  $P_2$ . What is the behavior of the two populations when  $c$  (harvesting rate of weasel) is varied? What is the value of  $c$  so that both populations are about the same?

```
r=5.4; L=100; a=4; c = 2.5; b = 0.1*a;
c=.
PP1 = c/b;
PP2 = r/a*(1 - c/(b*L));
p1=Plot[{PP1,PP2},{c,0,2}, AxesLabel->{"harvest rate", "Populations"}]
```

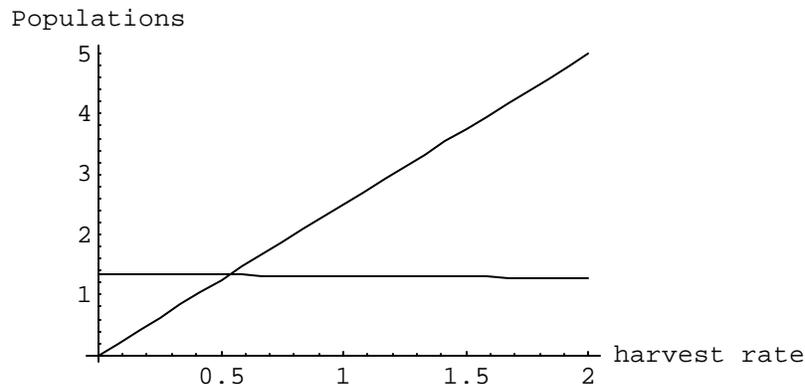


Figure 7.32. Harvest rate versus populations.

Harvest rate for populations to become equal

```
c=r/(a*(1/b+r/(a*b*L)));
prey = P1'[t] == r*P1[t] (1 - P1[t]/L) - a*P1[t]*P2[t];
pred = P2'[t] == - c*P2[t] + b*P1[t]*P2[t];
eq1 =NDSolve[{prey,pred, P1[0]==4, P2[0]==2}, {P1[t], P2[t]}, {t,0,50}];
p2=Plot [ Evaluate [{P1[t],P2[t]} /. eq1], {t,0,50},
  AxesLabel -> {"Years", "Voles and weasel pop."} ]
```

Voles and weasel pop.

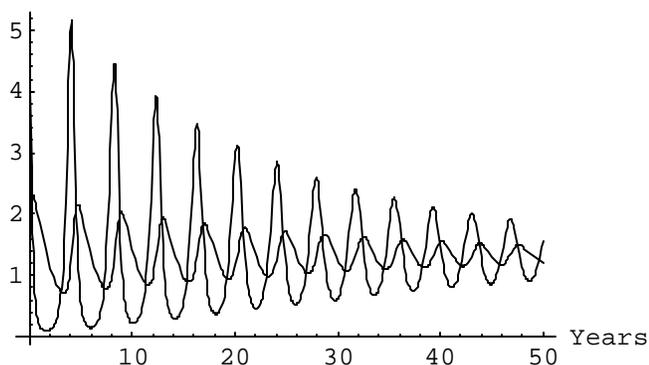


Figure 7.33. Interaction between voles and weasels at optimum harvest rate.

```

c=r/(a*(1/b+r/(a*b*L)));
prey = P1'[t] == r*P1[t] (1 - P1[t]/L) - a*P1[t]*P2[t];
pred = P2'[t] == - c*P2[t] + b*P1[t]*P2[t];
eq1 =NDSolve[{prey,pred, P1[0]==4, P2[0]==2}, {P1[t], P2[t]}, {t,0,50}];
p3=ParametricPlot[ {P1[t], P2[t]} /. eq1, {t,0,50}, PlotRange->{{0,6},{0,4}},
  Compiled -> False, AxesLabel -> {"Voles", "Weasels"}]

```

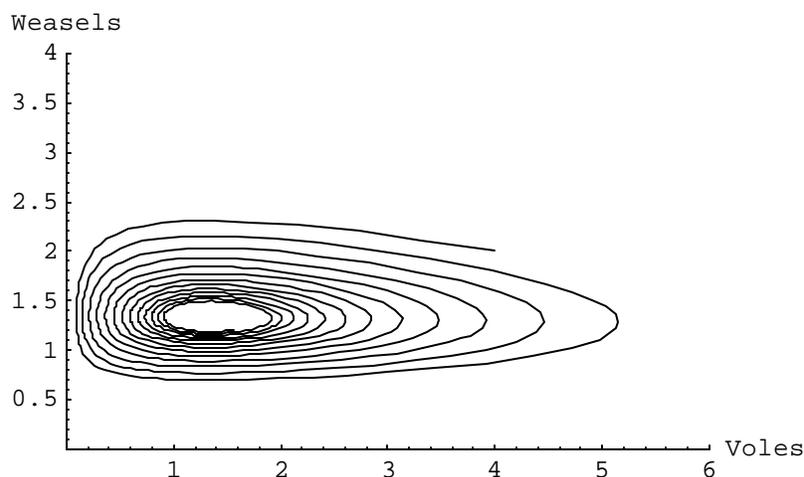


Figure 7.34. Phase-plane analysis between voles and weasels at optimum harvest rate.

By controlling the harvesting rate of  $P_2$ , we can cause  $P_1$  to equal  $P_2$  at equilibrium. Harvesting could be done through hunting or transfer of population out of the area.

## 7.4 Nutrient Loading

Consider a problem where an organic nutrient ( $P_1$ ), such as phosphates from detergents, fertilizers, and animal manure, enter a bay at a constant rate  $s$ . Small floating plant organisms, such as algae ( $P_2$ ), feed on the nutrient and the tiny crustaceans ( $P_3$ ) graze on the algae (Beltrami, 1987, p. 80). Because of the movement of the tides, a fraction  $q$  per unit time of the nutrients and algae are swept out of the ocean, never to return. Use the following equations to estimate the levels of nutrients, algae, and crustaceans over time.

$$\begin{aligned}\frac{dP_1}{dt} &= s - qP_1 - \frac{P_1 P_2}{a + P_1} \\ \frac{dP_2}{dt} &= -qP_2 + \frac{P_1 P_2}{a + P_1} - \frac{P_2 P_3}{b + P_2} \\ \frac{dP_3}{dt} &= -qP_3 + \frac{P_2 P_3}{b + P_2}\end{aligned}\quad (7.10)$$

where  $P_1$  is the amount of organic phosphates, in tons;  $P_2$  is the mass of algae, in tons;  $P_3$  is the mass of crustaceans, in tons;  $s$  is the constant rate of entry of organic phosphates, in tons of nutrients  $\text{yr}^{-1}$ ;  $q$  is the fraction of nutrients, algae and crustaceans that are swept out to the ocean; and the variables  $a$  and  $b$  are the predation factor, in tons.

Let the initial values be given as follows:  $s = 600$ ;  $q = 0.000001$ ;  $a = 100$ ;  $b = 400$ . Then the solution, using Mathematica, is as follows:

```
s = 60; q = 0.01; a = 1000; b = 4000;
eq1 = P1'[t] == s - q P1[t] - (P1[t] P2[t])/(a+P1[t]);
eq2 = P2'[t] == -q P2[t] + (P1[t] P2[t])/(a+P1[t]) - (P2[t] P3[t])/(b+P2[t]);
eq3 = P3'[t] == -q P3[t] + (P2[t] P3[t])/(b+P2[t]);
eq4 = NDSolve[{eq1, eq2, eq3, P1[0] == 800, P2[0] == 300, P3[0] == 50}, {P1[t], P2[t], P3[t]},
{t, 0, 500}, MaxSteps -> 2000];
p4 = Plot [ Evaluate [P1[t] /. eq4], {t, 0, 500}, PlotRange -> {{0, 500}, {0, 4000}}, AxesLabel ->
{"Time", "Nutrients"}];
```

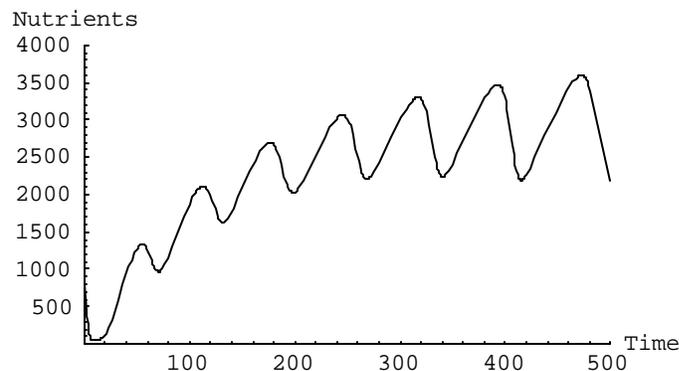


Figure 7.35. Pollutant accumulation over time.

```
p5=Plot [ Evaluate [P2[t] /. eq4], {t,0,500}, PlotRange->{{0,500},{0,4000}}, AxesLabel -> {"Time", "Algae"}];
```

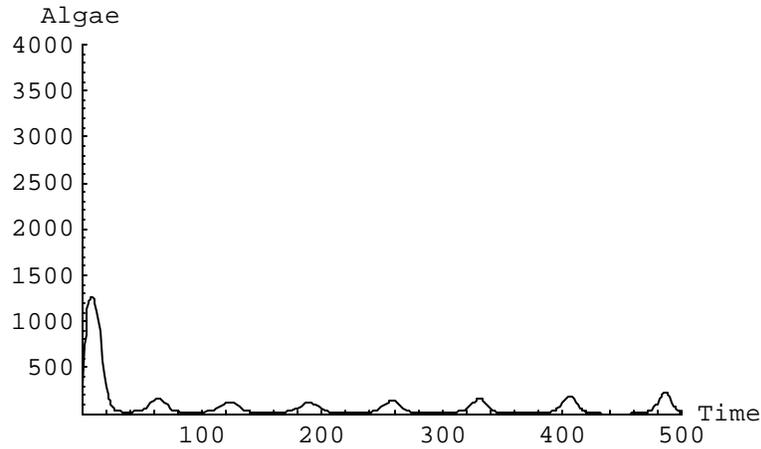


Figure 7.36. Algae growth over time.

```
p6=Plot [ Evaluate [P3[t] /. eq4], {t,0,500}, PlotRange->{{0,500},{0,4000}}, AxesLabel -> {"Time", "Crustaceans"}];
```

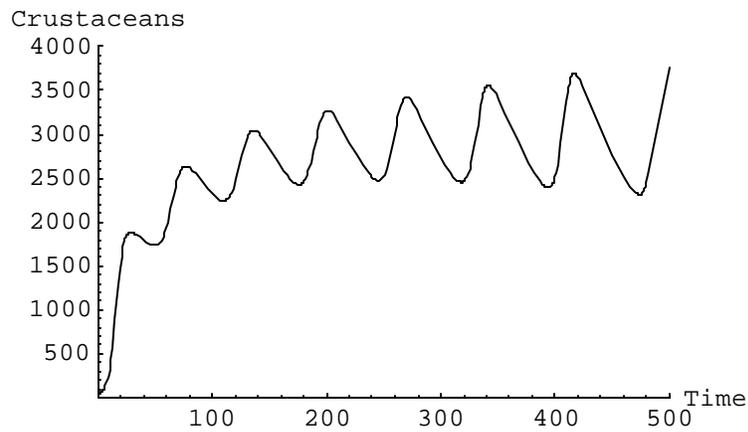


Figure 7.37. Crustacean growth over time.

```
Show[p4, p5, p6]
```

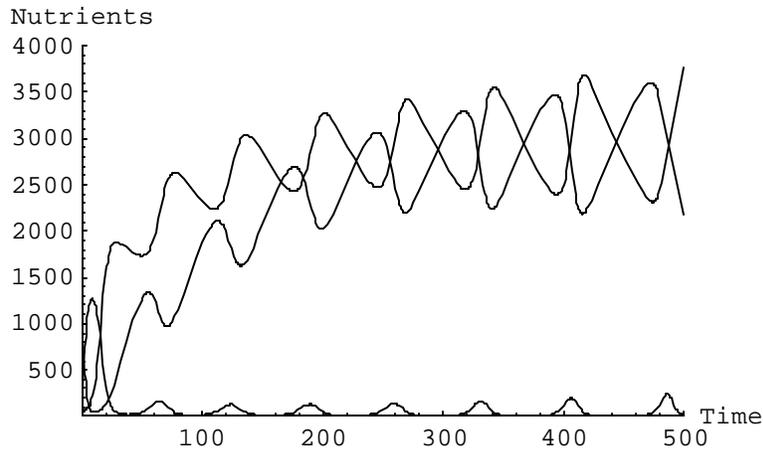


Figure 7.38. Interaction among nutrients, algae, and crustaceans over time.

*In[505]:=*

```
s = 60; q = 0.01; a = 1000; b = 4000;
eq1 = P1'[t] == s - q P1[t] - (P1[t] P2[t])/(a+P1[t]);
eq2 = P2'[t] == -q P2[t] + (P1[t] P2[t])/(a+P1[t]) - (P2[t] P3[t])/(b+P2[t]);
eq3 = P3'[t] == -q P3[t] + (P2[t] P3[t])/(b+P2[t]);
eq4 = NDSolve[{eq1, eq2, eq3, P1[0]==700, P2[0]==300, P3[0]==50}, {P1[t], P2[t], P3[t]},
{t, 0, 500}, MaxSteps -> 2000];
p7 = ParametricPlot[ {P1[t], P2[t]} /. eq4, {t, 0, 500}, Compiled -> False, PlotRange ->
>{{0, 4000}, {0, 1500}}, AxesLabel -> {"Nutrients", "Algae"}]
```

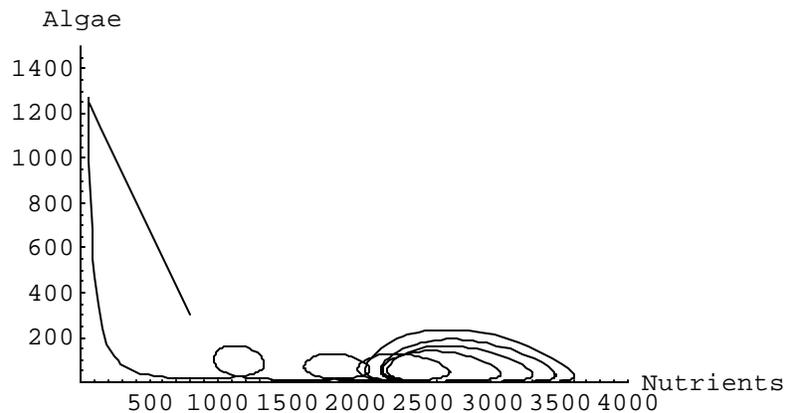


Figure 7.39. Phase-plane analysis between nutrients and algae.

```
p8 = ParametricPlot[ {P1[t], P3[t]} /. eq4, {t, 0, 500}, Compiled -> False, PlotRange ->
>{{0, 4000}, {0, 4000}}, AxesLabel -> {"Nutrients", "Crustaceans"}]
```

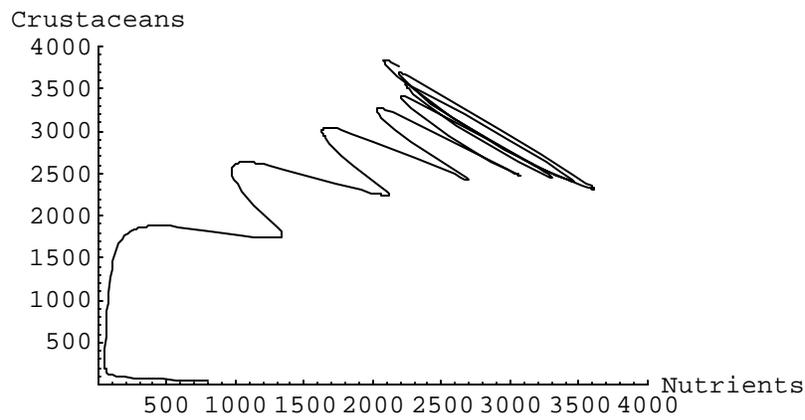


Figure 7.40. Phase-plane analysis between nutrients and crustaceans.

```
p9=ParametricPlot[ {P2[t], P3[t]} /. eq4, {t,0,500}, Compiled -> False, PlotRange->{{0,2000},{0,4000}}, AxesLabel -> {"Algae", "Crustaceans"}]
```

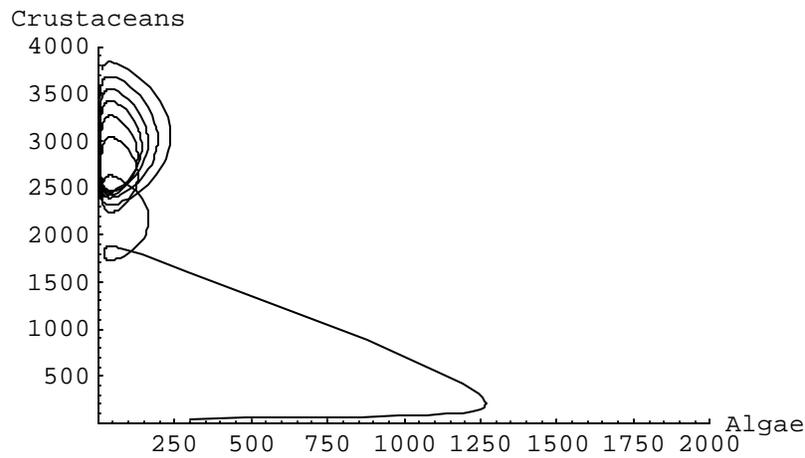


Figure 7.41. Phase-plane analysis between nutrients and crustaceans.

```
Show[{p7, p8}, PlotRange->{{0,4000},{0,4000}}, AxesLabel -> {"Nutrients", "Algae & Crustaceans"}]
```

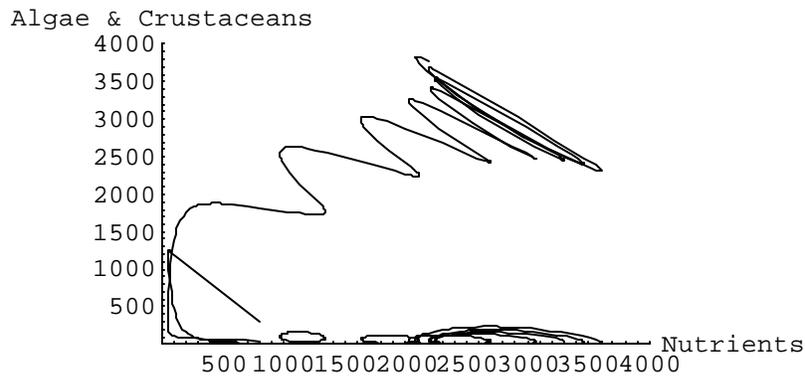


Figure 7.42. Phase-plane combination of nutrients, algae, and crustaceans.

## Exercises

1. If you were to manage the fishery project, what should be the initial fishery stock,  $P(0)$ , and initial harvesting effort,  $Ef(0)$ , so that the fish population can be maintained at steady state, given the original values of the parameters?
2. Another way of evaluating the fish harvesting dynamics is through the following interaction:

$$\frac{dP}{dt} = rP \left( 1 - \frac{P}{ak} \right) - \frac{q}{a} EP$$

$$\frac{DE}{dt} = eE \left( p \frac{q}{a} P - c \right)$$

Initially let  $r=100$ ,  $e=1$ ,  $c=100$ ,  $q=1$ ,  $p=1$ ,  $k=1000$ , and  $a=100$ . Run the model following the steps involved in the section 8.2 (Fishery Management). The Mathematica program is given below:

```
pop1[x_]:=Module[{eq3,P,Ef,t,eq1,eq2},
r=100; e=1; c=100; q=1; p=1; k=1000; a=x;
eq1=P'[t]==r P[t] (1 - P[t]/(k a)) - q/a Ef[t] P[t];
eq2=Ef'[t]==e Ef[t] (p q/a P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P[t], Ef[t]}, {t,0,1}, MaxSteps->2000];
Plot[Evaluate[{P[t]}/.eq3], {t,0,1}, DisplayFunction->Identity]
graphs=Table[pop1[t],{t,50,100,50}]
Show[graphs, PlotRange->{0,60000}, AxesLabel->{"Time","Fish
Stock"},DisplayFunction->$DisplayFunction]
```

```

pop2[x_]:=Module[ {eq3,P,Ef,t,eq1,eq2},
r=100; e=1; c=100; q=1; p=1; k=1000; a=x;
eq1=P'[t]== r P[t] (1 - P[t]/(k a)) - q/a Ef[t] P[t];
eq2=Ef[t] == e Ef[t] (p q/a P[t] - c);
eq3=NDSolve[{eq1, eq2, P[0]==100, Ef[0]==10}, {P[t], Ef[t]}, {t,0,1}, MaxSteps ->2000];
ParametricPlot [{P[t], Ef[t]} /. eq3, {t,0,1}, Compiled->False,
  DisplayFunction->Identity ]
graphs=Table[pop2[t],{t,50,100,50}]
Show[graphs, PlotRange -> {{0,80000},{0,80000}}, AxesLabel->{"Fish",
  "Harvesting Effort"},DisplayFunction->$DisplayFunction]

```

## Reference

Beltrami, E. 1987. Mathematics for Dynamic Modeling. Academic Press, Boston.